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Three-Dimensional Homogeneous (k, μ)-Contact Manifolds

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Abstract: In the present study, three-dimensional homogeneous Riemannian manifolds equipped with a left-invariant metric, which satisfy the (k, μ) condition have been considered. We bring a rich class of non-Sasakian manifolds which are (k, μ)-contact.

Keywords: Contact structure, (k, μ)-space, Homogeneous manifold, Unimodular and Non-unimodular.

1 Introduction

An almost contact structure on a $(2n + 1)$-dimensional smooth manifold M is a triplet (φ, ξ, η), where φ is a $(1, 1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\begin{aligned} \varphi(\xi) &= 0, \qquad \eta \circ \varphi = 0, \\ \eta(\xi) &= 1, \qquad \varphi^2 = -Id + \eta \otimes \xi, \end{aligned} \tag{1}$$

and φ has rank $2n$. A Riemannian metric g on M is said to be *compatible* with the almost contact structure (φ, ξ, η) if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

A smooth manifold M, equipped with an almost contact structure (φ, ξ, η) and a compatible Riemannian metric g, will be called an *almost contact metric manifold*. Remark that, by (1) and (2)

$$\eta(X) = g(\xi, X),$$

for any compatible metric.

A $(2n + 1)$-dimensional differentiable manifold M^{2n+1} is called a *contact manifold* if there exists a globally defined 1-form η such that $(d\eta)^n \wedge \eta \neq 0$. On a contact manifold there exists a unique global vector field ξ satisfying

$$d\eta(\xi, X) = 0, \qquad \eta(\xi) = 1,$$

for all $X \in TM^{2n+1}$.
Next, if the compatible Riemannian metric g satisfies

$$d\eta(X, Y) = g(X, \varphi Y),$$

then η is a contact form on M, ξ the associated Reeb vector field, g an associated metric, and

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$(M, \varphi, \xi, \eta, g)$ is called a contact metric manifold.

Denoting by $\mathcal{L}$ and R Lie differentiation and the curvature tensor respectively, we define on M^{2n+1} the $(1,1)$-tensor field h by

$$h = \frac{1}{2}\mathcal{L}_\xi \varphi. \tag{3}$$

The tensor field h satisfy

$$h\xi = 0, \qquad h\varphi = -\varphi h.$$

In this paper we recall some basic formula and discuss which case of the classification of three-dimensional homogeneous contact manifolds satisfies in (k, μ) condition, then the main interest is in the non-Sasakian contact metric manifolds with this curvature property.

2 Preliminaries

Let $(M^{2n+1}, \varphi, \eta, \xi, g)$ be a contact metric manifold. Consider

$$\begin{aligned} R(X,Y)\xi =& k(\eta(Y)X - \eta(X)Y) \\ &+ \mu(\eta(Y)hX - \eta(X)hY), \end{aligned} \tag{4}$$

for constants k and μ. a contact metric manifold satisfying this condition is called a (k, μ)-*manifold*. If k and μ be smooth functions, we call M a generalized (k, μ)-manifold.

If on a (k, μ)-manifold, $\mu = 0$, the contact metric manifold is said to be one for which ξ belongs to the k-nullity distribution.

Clearly, Sasakian spaces belong to this class ($k = 1$ and $h = 0$), $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$.

3 Homogeneous (k, μ)-manifold

Let G denote a three-dimensional Lie group and g a left-invariant Riemannian metric on G. As shown in [5], G is unimodular if and only if the linear map

$$L(x \times y) = [x, y], \;\; x, y \in g,$$

is self-adjoint, where $\times$ denotes the cross product operation. Consequently, if (G, g) is a three-dimensional unimodular Riemannian Lie group, then its Lie algebra g admits an orthonormal basis $\{e_1, e_2, e_3\}$, such that

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad , [e_3, e_1] = \lambda_2 e_2.$$

Let (φ, ξ, η) be a left-invariant almost contact metric structure on a unimodular Riemannian Lie group (G, g). The vector field ξ is a unit geodesic vector. Geodesic vector fields on three-dimensional unimodular Riemannian Lie groups were classified in [4]. The possible cases are the following.

(i) If all λ_i are distinct, then the only geodesic unit vector fields are $\pm e_i, i = 1, 2, 3$ (see Theorem 3.1 in [4]).

(ii) If at least two among λ_1, λ_2 and λ_3 coincide, then all unit vector fields in g are geodesic (see Remark in [4]). Indeed, such Riemannian Lie groups are naturally reductive (Theorem 6.5 of [7]) and so, all their geodesics are homogeneous.

Between the two cases above, just the first case will be a (k, μ)-space.

Case (i). Let (φ, ξ, η, g) be a left-invariant almost contact metric structure. Then, $\xi = \pm e_i$, for some $i = 1, 2, 3$. Thus, up to isomorphisms the general form of a left-invariant contact metric structure is given by

$$[\xi, e] = \lambda_{\sigma_3}\varphi e, \quad [e, \varphi e] = \lambda_{\sigma_1}\xi, \quad , [\varphi e, \xi] = \lambda_{\sigma_2} e, \tag{5}$$

where σ is a permutation of indices 1, 2, 3.

According to Koszul formula we have:

$$\begin{aligned} \nabla_e \xi &= \frac{\lambda_{\sigma_2} - \lambda_{\sigma_1} - \lambda_{\sigma_3}}{2}\varphi e, \\ \nabla_\xi e &= \frac{\lambda_{\sigma_2} - \lambda_{\sigma_1} + \lambda_{\sigma_3}}{2}\varphi e, \\ \nabla_\xi \xi &= 0, \quad \nabla_\xi \varphi e = \frac{\lambda_{\sigma_2} + \lambda_{\sigma_3} - \lambda_{\sigma_1}}{2} e, \\ \nabla_{\varphi e}\xi &= \frac{\lambda_{\sigma_1} + \lambda_{\sigma_2} - \lambda_{\sigma_3}}{2} e. \end{aligned}$$

By computing R we have:

$$\begin{aligned} R(e,\xi)\xi =&(\frac{\lambda_{\sigma_3}\lambda_{\sigma_1}+\lambda_{\sigma_3}\lambda_{\sigma_2}-\lambda_{\sigma_1}\lambda_{\sigma_2}}{2} \\ &+\frac{\lambda_{\sigma_2}^2+\lambda_{\sigma_1}^2-3\lambda_{\sigma_3}^2}{4})e, \\ R(\varphi e,\xi)\xi =&(\frac{\lambda_{\sigma_1}\lambda_{\sigma_2}+\lambda_{\sigma_2}\lambda_{\sigma_3}-\lambda_{\sigma_1}\lambda_{\sigma_3}}{2} \\ &+\frac{{\lambda_{\sigma_1}}^2+{\lambda_{\sigma_3}}^2-3{\lambda_{\sigma_2}}^2}{4})\varphi e, \\ R(e,\varphi e)\xi =&0, \end{aligned}$$

on the other hand by (3):

$$he=\frac{\lambda_{\sigma_3}-\lambda_{\sigma_2}}{2}e, \qquad h\varphi e=\frac{\lambda_{\delta_2}-\lambda_{\delta_3}}{2}\varphi e.$$

Finally we deduce:

$$\begin{aligned} k &= \frac{\lambda_{\sigma_1}^2-\lambda_{\sigma_2}^2-\lambda_{\sigma_3}^2+2\lambda_{\sigma_3}\lambda_{\sigma_3}}{4}, \\ \mu &= \frac{\lambda_{\sigma_3}^2-\lambda_{\sigma_2}^2+\lambda_{\sigma_1}\lambda_{\sigma_2}-\lambda_{\sigma_3}\lambda_{\sigma_1}}{\lambda_{\sigma_2}-\lambda_{\sigma_3}}, \end{aligned}$$

so this case is a (k,μ)-space.

Theorem 3.1. *Let (φ,ξ,η,g) be a left-invariant almost contact metric structure on a three-dimensional unimodular Lie group, as described by (5). Then*

the following properties are equivalent:

(i) $\lambda_{\sigma_2}=\lambda_{\sigma_3}$;
(ii) (φ,ξ,η,g) is normal;
(iii) (φ,ξ,η,g) is trans-Sasakian. In this case,$\bar{\alpha}=\frac{\lambda_{\sigma_1}}{2}$, $\bar{\beta}=0$. Thus, the structure is α-Sasakian, and cosymplectic if and only if $\lambda_{\sigma_1}=0$;
(iv) (φ,ξ,η,g) is quasi-Sasakian;
(v) ξ is a Killing vector field, that is,$\mathcal{L}_\xi g=0$, where$\mathcal{L}$ denotes the Lie derivative;
(vi) $\mathcal{L}_\xi\varphi=0$;
(vii) Lie group is a (k,μ)-space.

Proof. It is shown by a straightforward calculation that, if $\lambda_{\sigma_2}=\lambda_{\sigma_3}$, we have the necessary condition for manifold to be a (k,μ)-manifod with $k=\frac{1}{4}\lambda_{\sigma_2}^2$ and $\mu=$ arbitrary. So it is clear by [2] that the five next conditions are satisfied. □

According to classification of three-dimensional non-unimodular case in [2] and review them, just the second case is (k,μ)-space.

In this case from Milnor's classification of three-dimensional non-unimodular Riemannian Lie groups obtained in [5], If G is such a group
$\xi=\cos\theta e_2+\sin\theta e_3$.
In this case, an orthonormal basis (and so, a φ-basis) of ker η is given by
$\{E_1:=e_1, E_2:=-\sin\theta e_2+\cos\theta e_3\}$. We then easily obtain [2]:

$$\begin{aligned} [\xi,E_1] =&(\beta cos^2\theta+(\delta-\alpha)sin\theta cos\theta-\gamma sin^2\theta)E_2 \\ &-(\alpha cos^2\theta+(\beta+\gamma)sin\theta cos\theta+\delta sin^2\theta)\xi, \\ [\xi,E_2] =&0, \\ [E_1,E_2] =&(\delta cos^2\theta-(\beta+\gamma)sin\theta cos\theta+\alpha sin^2\theta)E_2 \\ &-(\gamma cos^2\theta+(\delta-\alpha)sin\theta cos\theta-\beta sin^2\theta)\xi. \end{aligned} \tag{6}$$

From [2] we must have $[\xi,E_1]\in ker\eta$, that is,

$$\alpha cos^2\theta+(\beta+\gamma)sin\theta cos\theta+\delta sin^2\theta=0. \tag{7}$$

We put

$$\begin{aligned} A &:= \delta cos^2\theta-(\beta+\gamma)sin\theta cos\theta+\alpha sin^2\theta, \\ B &:= -(\gamma cos^2\theta+(\delta-\alpha)sin\theta cos\theta-\beta sin^2\theta), \\ C &:= \beta cos^2\theta+(\delta-\alpha)sin\theta cos\theta-\gamma sin^2\theta. \end{aligned}$$

As usual, up to isomorphisms we can take $e=E_1$, $\varphi e=E_2$. Then, (φ,ξ,η) is described by

$$[\xi,e]=C\varphi e, \qquad [\xi,\varphi e]=0, \qquad [e,\varphi e]=A\varphi e+B\xi,$$

with $A\neq 0$.

In this case, η is a contact form if and only if $B\neq 0$.

By Calculating Curvature R, we obtain:

$$\begin{aligned} R(e,\xi)\xi &= (\frac{1}{4}B^2-\frac{3}{4}C^2+\frac{1}{2}BC)e, \\ R(\varphi e,\xi)\xi &= \frac{1}{4}(C-B)^2\varphi e, \\ R(e,\varphi e)\xi &= ACe, \end{aligned}$$

which implies $k = \frac{B^2}{4}$, $\mu = arbitrary$, $C = 0$ and $B \neq 0$.
So we have the following Theorem.

Theorem 3.2. *Let (φ, ξ, η, g) be a left-invariant almost contact metric structure on a three-dimensional non-unimodular Lie group, as described by (6). Then the following properties are equivalent:*

(i) $C = 0$;
(ii) (φ, ξ, η, g) is normal;
(iii) (φ, ξ, η, g) is quasi-Sasakian;
(iv)ξ is a Killing vector field;
(v) $\mathcal{L}_\xi \varphi = 0$;
(vi)In this case, Lie group is (k, μ)-manifold.

3.1 Non-Sasakian case

If (M, η, g) is a simply connected three-dimensional homogeneous contact Riemannian manifold, then $M = G$ is a Lie group and the contact metric structure (η, g, ξ, φ) is left-invariant [3].

According to classification of three-dimensional homogeneous contact Riemannian structures, in Non-Sasakian case, if G is non-unimodular, its Lie algebra is given by
$[e_1, e_2] = \alpha e_2 + 2\xi, \quad [e_1, \xi] = \gamma e_2, \quad [e_2, \xi] = 0,$
where $\alpha \neq 0$.

By Koszul formula we have:

$$\nabla_\xi e_1 = \frac{1}{2}(-2-\gamma)e_2, \qquad \nabla_{e_1}\xi = \frac{1}{2}(-2+\gamma)e_2,$$
$$\nabla_{e_2}\xi = \frac{1}{2}(2+\gamma)e_1, \qquad \nabla_\xi \xi = 0,$$
$$\nabla_\xi e_2 = \frac{1}{2}(2+\gamma)e_1, \qquad \nabla_{e_1} e_1 = 0,$$
$$\nabla_{e_2} e_2 = \alpha e_1,$$

and from (3), $he_1 = -\frac{1}{2}\gamma e_1, \; he_2 = \frac{1}{2}\gamma e_2$.

By curvature formula

$$R(e_1, \xi)\xi = (1 - \frac{3}{4}\gamma^2 - \gamma)e_1,$$
$$R(e_2, \xi)\xi = \frac{1}{4}(2+\gamma)^2 e_2,$$
$$R(e_1, e_2)\xi = -\alpha\gamma e_1,$$

from (4), G is a (k, μ)-manifold with $\gamma = 0, k = 1, \mu = arbitrary$.

References

[1] D.E. Blair, *Contact Metric Manifolds in Riemannian Geometry*, in: Lecture Notes in Math., vol. 509, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[2] G. Calvaruso, Three-dimensional homogeneous almost contact metric structures, Journal of Geometry and Physics 69 (2013) 6073

[3] G. Calvaruso, D. Perrone, Contact pseudo-metric manifolds, Differential Geom. Appl. 28 (2010) 615634.

[4] R.A. Marinosci, *Homogeneous geodesics in a three-dimensional Lie group, Comment.* Math. Univ. Carolin. **43** (2002) 261270.

[5] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Adv. Math. **21** (1976) 293329.

Books:

[6] D.E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, second ed., in: Progress in Math., vol. 203, Birkhuser, Boston, 2010.

[7] F. Tricerri, L. Vanhecke, *Homogeneous Structures on Riemannian Manifolds*, in: London Math. Soc. Lecture Notes Series, vol. 83, Cambridge University Press, 1983.

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On r-stable spacelike hypersurfaces in Lorentzian hyperbolic spaces

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Abstract: The notion of stability for some well known hypersurfaces of Riemannian manifolds has been started by Barbosa, do Carmo and others ([2, 3]). First discussions were on hypersurfaces with constant mean curvature and it has been generated to ones that have a constant higher order mean curvature ([4]). Recently, Brasil and Colares has paid attention to an extension of stability to spaclike hypersurfaces of de Sitter spaces with constant $r-$th mean curvature. In this paper, we extend the notion of r-stability to spacelike hypersurfaces of the Lorentzian hyperbolic spaces with constant higher order mean curvature.
Keywords: $r-$th mean curvature, r-Stability, Spacelike hypersurface.
MSC(2010): Primary 53C42, sceondary 53C20, 53B30, 83C99.

1 INTRODUCTION

In decade 80 Barbosa and do Carmo proved that spheres are the only stable critical points of the area function associated to volume-preserving variations. These results has been followed by many people and extended to hypersurfaces with constant mean or scalar curvature. The natural extension of mean and scalar curvature for hypersurfaces of dimension n is the higher order mean curvature. Recently, some people have studied the stability of hypersurfaces with constant (specially zero) $r-$th mean curvature in Riemannian space forms as spheres and hyperbolic spaces ([2, 3, 4, 1, 5]). We study the notion of r-stability to spacelike hypersurfaces of the Robertson-Walker space-time emphasizing on the anti de Sitter space with considering some conditions on their higher order mean curvature.

Here, we recall some basic preliminaries from [1, 2, 5]. By R^m_p, we mean the vector space R^m with metric $< x, y >:= -\Sigma_{i=1}^p x_i y_i + \Sigma_{j>p} x_j y_j$. Especially, $R^m_0 = R^m$, and R^m_1 is the Minkowski space. For $r > 0$ and $q = 0, 1$,

$$S^{n+1}_q(r) = \{y \in R^{n+2}_q | < y, y >= r^2\}$$

denotes the sphere (for $q = 0$) and de Sitter space (for $q = 1$) of radius r and curvature $1/r^2$, and

$$H^{n+1}_q(-r) = \{y \in R^{n+2}_{q+1} | < y, y >= -r^2\}$$

denotes the hyperbolic space (for $q = 0$) and anti-de Sitter space (for $q = 1$) of radius r and curvature $-1/r^2$. The simply connected space form $\tilde{M}^{n+1}_q(c)$ of curvature c and index q is R^{n+1}_q for $c = 0$, $S^{n+1}_q = S^{n+1}_q(1)$ for $c = 1$ and $H^{n+1}_q =$

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$H_q^{n+1}(-1)$ for $c = -1$. When $q = 0$, we take a component of H_0^{n+1}. The Weingarten formula for a spacelike hypersurface $x : M^n \to \tilde{M}_q^{n+1}(c)$ is $\bar{\nabla}_V W = \nabla_V W - \epsilon < SV, W > \mathbf{N}$, for $V, W \in \chi(M)$ where, $\epsilon = 2q-1$, $q \in \{0,1\}$ and S is the shape operator of M associated to a unit normal vector field $\mathbf{N}$ on M with $< \mathbf{N}, \mathbf{N} >= -\epsilon$. Since M is spacelike, S can be diagonalized. Denote its eigenvalues (the principal curvatures of M) by the functions $\kappa_1, ..., \kappa_n$ on M, define the elementary symmetric function as $s_j := \sum_{1 \leq i_1 < ... < i_j \leq n} \kappa_{i_1} ... \kappa_{i_j}$ and the j-th mean curvature of M by $\binom{n}{j} H_j = (-\epsilon)^j s_j$. The hypersurface M^n in R_p^{n+1} is called *j-minimal* if its $(j+1)$th mean curvature H_{j+1} is identically zero.

In particular, $H_1 = -\epsilon(1/n)tr(S)$ and $\mathbf{H} = H_1 \mathbf{N}$ are respectively the mean curvature and the mean curvature vector of M. The relation between the scalar curvature of M and H_2 as $tr(Ric) = n(n-1)(c - \epsilon H_2)$. In general, H_j is extrinsic (respectively, intrinsic) when j is an odd (respectively, an even) number, since the sign of H_j depends on the chosen orientation only in the odd case.

Proposion 1.1. *Let $x : M^n \to \tilde{M}_q^{n+1}(c)$ (where $n \geq 2$) be a connected spacelike hypersurface isometrically immersed into an standard Riemannian or Lorentzian space form, $c \in \{-1, 0, 1\}$ and $q \in \{0, 1\}$. Let $\kappa_1, ..., \kappa_n$ be the principal curvatures of M and H_r be the r-mean curvature of M. Then we have:*

(i) For $0 < r < n$, $H_r^2 \geq H_{r-1} H_{r+1}$. If $r = 1$ or if $r > 1$ and $H_{r+1} \neq 0$, then the equality happens iff $\kappa_1 = ... = \kappa_n$;

(ii) $H_1 \geq (H_2)^{1/2} \geq ... \geq (H_k)^{1/k}$ if $H_i > 0$ for $i = 1, ..., k$.

For a spacelike hypersurface M in the space form $\tilde{M}_q^{n+1}(c)$, we introduce the Newton transformations $P_j : \chi(M) \to \chi(M)$, associated with the shape operator S, inductively by

$$P_0 = I, P_j = (-\epsilon)^j s_j I + \epsilon S \circ P_{j-1} (j = 1, ..., n),$$

where I is the identity on $\chi(M)$. It can be seen that P_j has an explicit formula $P_j = (-\epsilon)^j \Sigma_{l=0}^j (-1)^l s_{j-l} S^l = \sum_{l=0}^j \binom{n}{j-l} \epsilon^l H_{j-l} S^l$, where, $H_0 = 1$ and $S^0 = I$. According to the characteristic polynomial of S, $Q_S(t) = \det(tI - S) = \sum_{l=0}^n (-1)^{n-l} s_{n-l} t^l$, the Cayley-Hamilton theorem gives $P_n = 0$.

Let $e_1, ..., e_n$ be a local orthonormal tangent frame on M that diagonalizes S and P_j as $Se_i = \kappa_i e_i$ and $P_j e_i = \mu_{i,j} e_i$, for $i = 1, 2, ..., n$, where $\mu_{i,j} = (-\epsilon)^j \Sigma_{i_1 < ... < i_j, i_l \neq i} \kappa_{i_1} ... \kappa_{i_j}$, (for $j = 0, 1, ..., n-1$). Using this and the useful identity

$$\epsilon \kappa_i \mu_{i,j} = \mu_{i,j+1} - (-\epsilon)^{j+1} s_{j+1} = \mu_{i,j+1} - \binom{n}{j+1} H_{j+1}, \quad (1)$$

and the notation $c_j = (n-j)\binom{n}{j} = (j+1)\binom{n}{j+1}$, the following properties of P_k may be obtained easily:

$$tr(P_j) = (-\epsilon)^j (n-j) s_j = c_j H_j, \quad (2)$$

$$tr(S \circ P_j) = (-\epsilon)^j (j+1) s_{j+1} = -\epsilon c_j H_{j+1}, \quad (3)$$

$$tr(S^2 \circ P_j) = \binom{n}{j+1} [n H_1 H_{j+1} - (n-j-1) H_{j+2}], \quad (4)$$

$$tr(P_j \circ \nabla_X S) = -\epsilon \binom{n}{j+1} < grad(H_{j+1}), X > . \quad (5)$$

Proposion 1.2. *Let $x : M^n \to \tilde{M}_q^{n+1}(c)$ (where $n \geq 2$) be a connected spacelike hypersurface isometrically immersed into an standard Riemannian or Lorentzian space form, $c \in \{-1, 0, 1\}$ and $q \in \{0, 1\}$. Let $\{e_1, ..., e_n\}$ and $\kappa_1, ..., \kappa_n$ be as in Proposition 1.1, and P_k be the $k-$th Newton map. If at a point $p \in M$, $H_k(p) = 0$ and $H_{k+1}(p) \neq 0$, then P_{k-1} is definite at p.*

Now, we define the notion of variation, the linearized operator L_j, r-stability and the index of r-stability.

Definition 1.3. *Let $x : M^n \to \tilde{M}_q^{n+1}(c)$ be a compact connected orientable spacelike hypersurface isometrically immersed into an standard Riemannian or Lorentzian space form, $c \in \{-1, 0, 1\}$ and $q \in \{0, 1\}$. A map $X : (-\epsilon, \epsilon) \times M^n \to \tilde{M}_q^{n+1}(c)$ is called a variation of M^n if it satisfies the following properties:*

(1) *For each $t \in (-\epsilon, \epsilon)$ the map $X_t : M^n \to \tilde{M}_q^{n+1}(c)$ by rule $X_t(p) := X(t, p)$, is an immersion.*

(2) $X_0 = x$ *and for every* $t \in (-\epsilon, \epsilon)$, $X_t|_b d(M) = x|_b d(M)$.

Definition 1.4. *The linearized operator of the* $(j+1)$*-th mean curvature of* M, $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ *is defined by the formula* $L_j(f) := tr(P_j \circ \nabla^2 f)$, *where* $\nabla^2 f$ *is given by* $< \nabla^2 f(X), Y >= Hess(f)(X,Y)$.

Among many interesting properties of L_j, we point that for a normal variation of M with variational field $\frac{dX_t}{dt}(t)|_{t=0} = f\mathbf{N}$, we have the equality
$\frac{d}{dt} s_{j+1}|_{t=0} = L_j f + (s_1 s_{j+1} - (j+2)s_{j+2})f + c(n-j)s_j f$,
where L_j is the principal part of the linearized operator associated to s_{j+1}. For convenience, we define the operator J_j as:
$J_j := L_j + (s_1 s_{j+1} - (j+2)s_{j+2})I + c(n-j)s_j I$,
as well as a bilinear symmetric form B_j can be defined by $B_j(f,g) := -\int_M g J_j f dM$.

Definition 1.5. *Let* $x : M^n \to \bar{M}_q^{n+1}(c)$ *be as in Definition* 1.3 *with condition that* H_{r+1} *is constant.* M^n *is called* r*-stable if* $B_r(f,f) \geq 0$ *for all* $f \in \mathcal{C}_c^\infty(M)$.

2 Main results

Lemma 2.1. *Let* $x : M^n \to H_1^{n+1}$ *be a connected orientable spacelike hypersurface isometrically immersed into the Lorentzian hyperbolic space and* $X : M^n \times (-\epsilon, \epsilon) \to H_1^{n+1}$ *be a normal variation of* x. *Then we have* $\frac{\partial s_{k+1}}{\partial t} = (-1)^{k+1}(L_k f - tr(P_k)f - tr(S^2 P_r)f)$.

Lemma 2.2. *Let* $x : M^n \to H_1^{n+1}$ *be a connected orientable spacelike hypersurface isometrically immersed into the Lorentzian hyperbolic space and* $X : M^n \times (-\epsilon, \epsilon) \to H_1^{n+1}$ *be a normal variation of* x. *Then we have* $\frac{\partial s_{k+1}}{\partial t} = (-1)^{k+1}(L_k f - tr(P_k)f - tr(S^2 P_r)f)$.

Here is some of our results on the stability and index of r-stability of spacelike hypersurfaces in the standard Lorentzian space forms.

Theorem 2.3. *Let* $x : M^n \to H_1^{n+1}$ *be a connected time-oriented spacelike hypersurface isometrically immersed into anti-de Sitter space. Then, the following statements are equivalent:*
(1) x *has constant scalar curvature;*
(2) For all volume-preserving variations, $\frac{d}{dt}\int_M nH_0(t)dM_t = 0$;
(3) For all variations $\frac{dJ(t)}{dt}|_{t=0} = 0$.

Theorem 2.4. *Let* $x : M^n \to H_1^{n+1}$ *be a connected spacelike hypersurface isometrically immersed into anti-de Sitter space with gauss map* $\boldsymbol{N}$. *Let* Y *be the variational vector field of* M *and* $g :< Y, \boldsymbol{N} >$. *Then* M *is stable if and only if* $\alpha := \frac{-1}{2}n^2(n-1)(1+R)H_0 - n(n-1)H - 3s_3$ *be an eigenvalue of* $L_1 g$.

Similar results have been gotten about the closed connected timelike hypersurfaces of the de Sitter space S_1^{n+1} and anti de Sitter space H_1^{n+1}.

References

[1] A. Barros, A. Brasil, A. Caminha, *stability of spacelike hypersurfaces in foliated spacetimes*, Diff. Geom. and Appl. 26 (2008), 357365.

[2] J.L.M. Barbosa, M. do Carmo *Stability of hypersurfaces with constant mean curvature*, Math. Z. 185 (1984) 339353.

[3] J.L.M. Barbosa, M. do Carmo, J. Eschenburg, *Stability of hypersurfaces with constant mean curvature*, Math. Z. 197 (1988) 123138.

[4] J.L.M. Barbosa, A.G. Colares, *Stability of hypersurfaces with constant* $r-$*mean curvature*, Ann. Global Anal. Geom. 15 (1997) 227297.

[5] A. Barsil, A.G. Colares, *Stability of spacelike hypersurfaces with constant* $r-$*mean curvature in de Sitter space*, Proc. of Geom. and Phys. 7 (2004) 139145.

[6] B. O'Neill, *Semi-Riemannian geometry with applicatins to relativity*, Acad. Press Inc. (1983).

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On the Image of Gauss Map

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Abstract: In this paper, we are supposed to investigate some properties of the integral curves of some vector fields X which their restriction to an $n-$surface S is tangent to S. Then as applications of the Theory of ODE, we obtain some necessary and sufficient conditions to determine the relationship between the shape of an $n-$surface S and the image of its Gauss map in the following cases: (i) The image is a point, (ii) The image lies in an $n-$plane $a_1x_1 + \cdots + a_{n+1}x_{n+1} = 0$, (iii) The image lies in an $n-$plane $x_1 = c$ for $0 < |c| < 1$, (iv) The image lies in an $n-$dimensional right circular cone. Finally we show that that although the image of the Gauss map of an $n-$surface, in general is not an $m-$surface for any $m \in N$, but for every open subset U of the $n-$dimensional unit sphere S_n, there exists an $n-$surface S whose spherical image is U.

Keywords: Applications to differential and integral equations, Shape Theory, Vector Field.

1 INTRODUCTION

By an $n-$surface in U (simply $n-$surface) we meant a nonempty set $S = f^{-1}(c)$ where $f : U \to R$ is a smooth map defined on an open set $U \subseteq R^{n+1}$ and $c \in R$ such that $\nabla f(q) \neq 0$ for all $q \in S$. An oriented $n-$surface is an $n-$surface S together with a smooth unit normal vector field $\mathbf{N}$ on S [1]. The function $N : S \to R^{n+1}$ associated with the vector field $\mathbf{N}$ by $\mathbf{N}(p) = (p, N(p))$ which maps S into the unit sphere $S_n \subset R^{n+1}$ is called the Gauss map [1]. A well known Theorem shows that if S be a compact connected oriented $n-$surface in R^{n+1} exhibited as a level set $f^{-1}(c)$ of a smooth function $f : R^{n+1} \to R$ with $\nabla f(p) \neq 0$ for all $p \in S$, then the Gauss map, maps S onto the unit sphere S_n [1]. Also it is proved that there exists a conformal non-minimal immersion from a simply-connected surface in R^{n+1} with a given Gauss map, if and only if a set of differential equations depending on the conformal structure and the Gauss map is satisfied [2]-[3]-[4]. The Gauss map of a minimal surface in the Euclidean space R^3 is a conformal map. This fact has deep consequences in the behavior of these surfaces and has allowed a massive presence of complex variable techniques in the classical theory of minimal surfaces. For example the Gauss map of a non-flat complete minimal surface must be dense in the unit sphere S_2[6]. Furthermore there are some inequalities which estimate the curvature of all minimal surfaces whose Gauss map omits a fixed geodesic disc in S_2 [7]. In this study we investigate some properties of the image of the Gauss map of $n-$surfaces as some applications of the theory of differential equations. The paper is based on the following Theorem which is a consequence of the fundamental theorem of existence and uniqueness of solutions of systems of first order differential equations and asserts the existence and uniqueness of integral curves on $n-$surfaces [5]-[9]. Some ap-

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plications of this Theorem for introducing a generalization of tangent vector fields on $n-$surfaces has done earlier [8].

Theorem 1.1. *[10] Let S be an $n-$surface, let X be a smooth tangent vector field on S and $p \in S$. Then there exists an open interval I containing 0 and a parameterized curve $\alpha : I \to S$ such that,*

1. *$\alpha(0) = p$,*
2. *$\dot{\alpha}(t) = X(\alpha(t))$ for all $t \in I$,*
3. *If $\beta : \tilde{I} \to S$ is any other parameterized curve in S satisfying (1) and (2), then $\tilde{I} \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.*

Theorem 1.2. *Let $S = f^{-1}(c)$ be an $n-$surface and X be a smooth vector field on U whose restriction to S is a tangent vector field on S. If $\alpha : I \to U$ is any integral curve of X such that $\alpha(t_0) \in S$ for some $t_0 \in I$, then $\alpha(t) \in S$ for all $t \in I$.*

2 MAIN RESULTS

Let $a = (a_1, ..., a_{n+1}) \in R^{n+1}, a \neq 0$. By an $n-$plane we meant the set

$$P = \{(x_1, ..., x_{n+1}) | a_1x_1 + \cdots + a_{n+1}x_{n+1} = c\}$$

for some $c \in R$. Evidently, the image of the Gauss map of an $n-$plane is a single point. In the following, we establish a converse for this fact.

Theorem 2.1. *Let the image of the Gauss map of an $n-$surface S be a single point v and B be an open ball which is contained in U. Let $p \in B \cap S$ and $H = \{x \in R^{n+1} | x \cdot v = p \cdot v\}$. Then $B \cap H \subseteq S$.*

Proof. Let $x_0 \in B \cap H$, $w = x_0 - p$ and consider the constant vector field $X(q) = (q, w)$. X is a smooth vector field on U whose restriction to S is tangent to S and its integral curve through x_0 is $\alpha(t) = (t + 1)x_0 - tp$ with $t \in I$ for some open interval I containing $\{-1, 0\}$. Since $\alpha(-1) = p$, then Theorem 1.2 asserts that $\alpha(t) \in S$ for all $t \in R$. Therefore $x_0 = \alpha(0) \in S$ and $B \cap H \subseteq S$. □

Theorem 2.2. *Let the image of the Gauss map of an arc wise connected $n-$surface S be a single point v. Let $p \in S$ and $H = \{x \in R^{n+1} | x \cdot v = p \cdot v\}$ be an $n-$plane through p. Then $S \subseteq H$.*

Proof. Let $x_0 \in S$ and $\alpha : [t_1, t_2] \to S$ be a continuous map such that $\alpha(t_1) = p, \alpha(t_2) = x_0$. Let $\alpha(t_1) \cdot v < \alpha(t_2) \cdot v$ and consider the open subset

$$O = \{x \in R^{n+1} | \alpha(t_1) \cdot v < x \cdot v < \alpha(t_2) \cdot v\}$$

Thus O is an infinite union of $n-$planes

$$H_{\alpha(t_0)} = \{x \in R^{n+1} | x \cdot v = \alpha(t_0) \cdot v\}$$

for some t_0 with $t_1 < t_0 < t_2$. Let $B \subseteq U$ is an open ball containing $\alpha(t_0)$ and $x_1 \in H_{\alpha(t_0)} \cap B$. Let X be the constant vector field $X(q) = (q, w)$ with $w = x_1 - \alpha(t_0)$ defined on B. A similar argument as in the proof of Theorem 2.1 shows that $x_1 \in S$. Therefore $O \cap B \subseteq S$ which is impossible. Similarly $\alpha(t_1) \cdot v > \alpha(t_2) \cdot v$ tends to a contradiction. □

Corollary 2.3. *The image of the Gauss map of an arc wise connected $n-$surface S is a single point if and only if S is a part of an $n-$plane.*

Theorem 2.4. *Let S be an $n-$surface, $a = (a_1, ..., a_{n+1}) \in R^{n+1}, a \neq 0$ and*

$$H = \{(x_1, ..., x_{n+1}) | a_1x_1 + \cdots + a_{n+1}x_{n+1} = 0\}$$

be an $n-$plane. If $p \in S$ and there is an open interval I about 0 such that $p + ta \in S$ for all $t \in I$, then the spherical image of S is contained in the $n-$plane H.

Proof. Let S be an $n-$surface such that for any $p \in S$ and an open interval I about 0, $p + ta \in S$ for all $t \in I$. Let N be the Gauss map of S. Then $N \perp \alpha'(t)$ for any smooth map $\alpha : I \to S$ and $t \in I$. Consider the map $\alpha(t) = p + ta$, then $\alpha'(t) = a$ and $N \cdot a = 0$, therefore $a_1N_1 + \cdots + a_{n+1}N_{n+1} = 0$ and N is contained in the $n-$plane H. □

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Theorem 2.5. *Let $a = (a_1, ..., a_{n+1}) \in R^{n+1}, a \neq 0$ and S be an $n-$surface. If the image of the Gauss map of S is contained in the $n-$plane*

$$H = \{(x_1, ..., x_{n+1}) | a_1x_1 + \cdots + a_{n+1}x_{n+1} = 0\}$$

Then there is an open interval I about 0 such that $p + ta \in S$ for all $t \in I$.

Proof. Let N, the image of the Gauss map of S is contained in the $n-$plane H, i.e., $a \cdot N = 0$. Consider the constant vector field $X(q) = (q, a)$ for $a = (a_1, ..., a_{n+1})$. The integral curve of X through $p \in S$ is $\alpha(t) = ta + p, t \in I$ for some open interval I about 0. The restriction of X to S is tangent to S and $\alpha(0) = p \in S$. So Theorem 1.2 implies that $\alpha(t) \in S$ for all $t \in I$. □

Corollary 2.6. *Let $a = (a_1, ..., a_{n+1}) \in R^{n+1}, a \neq 0$ and S be an $n-$surface. Then the image of the Gauss map of S is contained in the $n-$plane $H = \{(x_1, ..., x_{n+1}) | a_1x_1 + \cdots + a_{n+1}x_{n+1} = 0\}$, if and only if there exists an open interval I about 0 such that $p + ta \in S$ for all $t \in I$.*

Example 2.7. *Let $U = \{(x, y, z) \in R^3 || z| > 2\}$, $f : U \to R$ be defined by $f(x, y, z) = (y^2 - x^2 - z^2)(x^2 + y^2 - 1)$ and $S = f^{-1}(0)$. Then S is a non connected $2-$surface which is a union of two disjoint connected $2-$surfaces. The image of the Gauss map of S is the union of the circle*

$$(x^2 + y^2 - 1)^2 + z^2 = 0$$

and the two following disjoint circles

$$(x^2 + y^2 + z^2 - 1)^2 + (2y^2 - 1)^2 = 0$$

Since these circles intersect each other in the points $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0)$, the image of the Gauss map of an $n-$surface for some $n \in N$, however, generally not an $m-$surface for any $m \in N$.

Lemma 2.8. *For any open subset U of the $n-$sphere S_n, there exists an $n-$surface S whose image of its Gauss map is U.*

Proof. Let O be an open subset of R^{n+1} such that $U = O \cap S$. Clearly $S = f^{-1}(0)$ for f defined by $f(x_1, ..., x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2 - 1$. Define $g : U \to R$ by $g = f|_O$, then $U = g^{-1}(0)$ is an $n-$surface with the required property. □

3 APPLICATIONS

Let ℓ be the half line in R^2 with equation $x_2 = \rho x_1 + \theta$ for some $\rho, \theta \in R$ such that $x_2 > 0$. Define the function $f : R \times R^+ \times \overbrace{R \times \cdots \times R}^{(n-1) times} \to R$ by

$$f(x_1, x_2, ..., x_{n+1}) = \sqrt{x_2^2 + \cdots + x_{n+1}^2} - \rho x_1 - \theta$$

for $n \geq 2$. Let $S_\ell = f^{-1}(0)$ and

$$y_i = \frac{x_i}{\sqrt{\rho^2 + 1}\sqrt{x_2^2 + \cdots + x_{n+1}^2}}$$

for $2 \leq i \leq n + 1$. It can be seen that S_ℓ is an $n-$surface of revolution obtained by rotating ℓ about the x_1 axis with unit normal

$$N(x_1, x_2, ..., x_{n+1}) = (\frac{-\rho}{\sqrt{\rho^2 + 1}}, y_2, ..., y_{n+1})$$

So the image of the Gauss map of S_ℓ, i.e., the circle

$$x_2^2 + \cdots + x_{n+1}^2 = \frac{1}{\rho^2 + 1}, x_1 = \frac{-\rho}{\sqrt{\rho^2 + 1}}$$

lies in the plane $x_1 = \frac{-\rho}{\sqrt{\rho^2+1}}, 0 < \frac{|\rho|}{\sqrt{\rho^2+1}} < 1$. Conversely, let $\mathcal{C}$ be the graph of a smooth map $g : R \to R^+$ and $f : R \times R^+ \times \overbrace{R \times \cdots \times R}^{(n-1) times} \to R$ be defined by

$$f(x_1, x_2, ..., x_{n+1}) = \sqrt{x_2^2 + \cdots + x_{n+1}^2} - g(x_1)$$

Let $S_\mathcal{C} = f^{-1}(0)$ be the $n-$surface of revolution obtained by rotating $\mathcal{C}$ about the x_1 axis such that the image of the Gauss map of $S_\mathcal{C}$ lies in some plane $x_1 = b$ with $0 < |b| < 1$. Let $z_i = \frac{x_i}{\sqrt{x_2^2+...+x_{n+1}^2}}$ for $2 \leq i \leq n + 1$. Since the unit normal of $S_\mathcal{C}$ is

$$N(x_1, x_2, ..., x_{n+1}) = (\frac{-g'(x_1)}{\sqrt{(g'(x_1))^2 + 1}}, \frac{z_2}{\sqrt{(g'(x_1))^2 + 1}}, ..., \frac{z_{n+1}}{\sqrt{(g'(x_1))^2 + 1}})$$

it follows that $\frac{-g'(x_1)}{\sqrt{(g'(x_1))^2+1}} = b$ and hence $g'(x_1) = \frac{\pm b}{\sqrt{1-b^2}}$. Thus $g(x_1) = \rho x_1 + \theta$ for some constants $\rho, \theta \in R$. Also it can be seen that the image of the Gauss map of S_ℓ lies in the $n-$dimensional right circular cone $x_1^2 = \rho^2(x_2^2 + \cdots + x_{n+1}^2)$. Conversely if the image of the Gauss map of $S_\mathcal{C}$ lies in an $n-$dimensional right circular cone $x_1^2 = \rho^2(x_2^2+\cdots+x_{n+1}^2)$ with some constant $\rho \in R$, then a simple calculation shows that $g'(x_1) = \pm\rho$ and so $g(x_1) = \pm\rho x_1 + \theta$ for some $\theta \in R$.

Corollary 3.1. *Let $g : R \to R^+$ be a smooth map and $f : R \times R^+ \to R$ be defined by $f(x_1, x_2) = x_2 - g(x_1)$. Let the image of the Gauss map of the $n-$surface of revolution obtained by rotating the curve $\mathcal{C} = f^{-1}(0)$ about the x_1 axis is denoted by $IG(S_\mathcal{C})$. Then the following statements are equivalent:*

1. *$IG(S_\mathcal{C})$ lies in the plane $x_1 = b$ with $0 < |b| < 1$;*
2. *$IG(S_\mathcal{C})$ lies in the $n-$dimensional right circular cone $x_1^2 = \rho^2(x_2^2+\cdots+x_{n+1}^2)$ with some constant $\rho \in R$;*
3. *$S_\mathcal{C}$ is an $n-$dimensional right circular cone.*

Corollary 3.2. *With any smooth map $g : R \to R^+$ and $f : R\times R^+ \to R$ which is defined by $f(x_1, x_2) = x_2 - g(x_1)$, $IG(S_\mathcal{C})$ lies in the $n-$dimensional perforated sphere $S_n - \{\pm1, 0, \cdots, 0\}$.*

Corollary 3.3. *There exists an $n-$surface of revolution S whose image of the Gauss map is exactly $S_n - \{\pm1, 0, \cdots, 0\}$.*

References

[1] M. Do Carmo, Differential Geometry of Curves and Surfaces, Cambridge, Mass: M.I.T Press , 1958, p.136.

[2] D. Hoffman and R. Osserman, "On the Gauss map of surfaces in R^n," Math. Ann., 283, pp. 375-380, 1989.

[3] D. Hoffman and R. Osserman, "The Gauss map of surfaces in R^3 and R^4," Proc. London Math. Soc., 50, pp. 27-56, 1985.

[4] D. Hoffman, R. Osserman, and R. Schoen, "On the Gauss map of complete constant mean curvature surfaces in R^3 and R^4," Comm. Math. Helvetici, 57, pp. 519-531, 1982.

[5] W. Hurwics, Lectures on Ordinary Differential Equations, Englewood Cliffs: N.J, Prentice Hall , 1976, p. 28.

[6] R. Osserman, "Proof of a conjecture of Nirenberg," Comm. Pure Appl. Math., 12, pp. 229-232, 1959.

[7] R. Osserman, "A sharp Schwarz inequality on the boundary," Proc. Amer. Math. Soc., 128, pp. 3513-3517, 2000.

[8] A. Parsian, "On the Integral Curves of $\theta-$vector fields on surfaces," Presented at the fourth Mathematics Annual Nat. Conf. of Payame Noor University, Iran, 2011.

[9] A. Parsian, "Extended Decomposition Theorem and its Applications, Presented at the fourth Mathematics Annual Nat. Conf. of Payame Noor University, Iran, 2011.

[10] M. Spivak, A comprehensive introduction to differential Geometry, vol. I. Boston: 1970, p. 139.

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Long time convergence of a flow on Hessian manifolds

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Abstract: This paper is devoted to introduce a special flow on Hessian manifolds. We show that with a specific condition on curvature of Hessian manifolds, this flow has long time solution and converges to Hessian -Einsteinian metrics.
Keywords: Hessian manifolds, Flow, Kähler Ricci flow, Hessian-Einstein, Complete manifolds.

1 INTRODUCTION

A Riemannian metric on a flat manifold is called a Hessian metric if it is locally expressed by the Hessian of functions. Typical examples of these manifolds include regular convex cones, and the space of all positive definite real symmetric matrices. The first Hessian structures was studied by Koszul [4], then Vinberg consider these metrics on convex cone [8] and S.Y. Cheng and S.T. Yau use Hessian metrics to solve Monges-Ampere equations [3]. The geometry of Hessian manifolds is very similar and near to Kählerian manifolds. In fact, relations of these two kind of manifolds was motivation of defining a flow on Hessian manifolds similar to Kähler Ricci flow.
In the following, after some preliminaries we present a complete introduction to Hessian manifolds and define our flow and obtain some conditions on which the flow is convergent. Similar to other flows on Riemannain manifolds, this flow is an evolution equation which can be used in order to deform initial metric into a suitable metric that can specify the topology and geometry of underlying manifold.

2 Preliminaries

let M^n be an n-dimensional C^∞ manifold and TM denote the tangent bundle of M, with projection map $\pi\colon M \to TM$. If $(x^1, x^2, \cdots, x^n)$ are local coordinates on M, we set

$$q^i := x^i o\pi, \qquad q^{n+i} := dx^i,$$

then $(q^1, \cdots, q^n, q^{n+1}, \cdots, q^{2n})$ form local coordinates on 2n-dimensional manifold TM.

Definition 2.1. *Let X be a vector field on M, it's vertical lift X^V is a vector field on TM defined by*

$$X^V := \omega(X)o\pi \qquad \omega \in \mathcal{A}^1(M),$$

where ω is a 1-form on M, that is regarded as a function on TM here.

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Definition 2.2. *Let D be a linear connection on M, horizontal lift of vector field X, X^H is a vector field on TM defined by*

$$X^H(\omega) = D_X^{\omega}.$$

Remark 2.3. *The span of horizontal lifts at $t = (p, v) \in TM$ is called the horizontal subspace $\mathcal{H}_{(p,v)}$ of T_tTM at point (p, v).*

Remark 2.4. *The span of vertical lifts at $t = (p, v) \in TM$ is called the vertical subspace $\mathcal{V}_{(p,v)}$ of T_tTM at point (p, v).*

Definition 2.5. *If V be a neighborhood of $p \in M$ such that exp is a diffeomorphism of a neighborhood V' of 0 in T_pM onto V,(normal neighborhood), then $\tau\colon \pi^{-1}(V) \to T_pM$ be a C^∞ map which translate each $Y \in \pi^{-1}(V)$ in a parallel manner from $q = \pi(Y)$ to p along the unique geodesic in V from q to p, and for $u \in T_pM$ the R_{-u} be a map defined by*

$$R_{-u}\colon T_pM \longrightarrow T_pM,$$
$$v \to v - u,$$

the connection map

$$K_{(p,u)}\colon T_{(p,u)}TM \longrightarrow T_pM,$$

of connection D is defined by

$$K(X) := d(expo\tau oR_{-u})(X),$$

for all $X \in T_{(p,u)}TM$.

We show that the horizontal subspace $\mathcal{H}_{(p,u)}$ is the kernel of connection map at (p, u), i.e.

$$K(X^V) = 0, \qquad K(X^H) = X_\pi, \forall X \in T_{(p,u)}TM.$$

Theorem 2.6. *The tangent space $T_{(p,u)}TM$ of tangent bundle TM at point (p, u) is the direct sum of it's vertical and horizontal subspaces, i.e.*

$$T_{(p,u)}TM = \mathcal{H}_{(p,u)} \oplus \mathcal{V}_{(p,u)}.$$

[1].

Remark 2.7. *For each tangent vector $Z \in T_{(p,u)}TM$ we can write*

$$Z = X^H + X^V,$$

where X, Y are determined by $X = d\pi(Z)$ and $Y = K(Z)$.

2.1 The lifts in local coordinates

Let $(x^1, \cdots, x^n)$ and $(q^1, \cdots, q^n, q^{n+1} \cdots, q^{2n})$ be local coordinates on M and TM respectively. Then for a point $p \in M$, $\{\frac{\partial}{\partial x^i}\}_p$ for $i = 1, \cdots, n$ form a base for tangent space T_pM and$\{\frac{\partial}{\partial q^i}\}_{(p,u)}$ is a base for $T_{(p,u)}TM$ at point (p, u). Using local expressions for vertical and horizontal lifts we have

$$(\frac{\partial}{\partial x^i})^H = \frac{\partial}{\partial q^i} - \sum_{j,k=1}^{n} (\Gamma_{ij}^k o\pi) q^{n+j} \frac{\partial}{\partial q^{n+k}}, \qquad (1)$$

$$(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial q^{n+i}}. \qquad (2)$$

2.2 The almost complex structure on TM

Let $J\colon TTM \longrightarrow TTM$ be a linear endomorphism of the tangent bundle of TM characterized by

$$d\pi(JX) = -K(X), \qquad \text{and} \qquad K(JX) = d\pi(X),$$

for all $X \in \tau(M)$, so we have

$$J(X^V) = -X^H, \qquad J(X^H) = X^V,$$

therefore the endomorphism J satisfies

$$J^2 = -Id_{TTM},$$

hence it's an almost complex structure on TM.

Proposition 2.8. *Almost complex structure J on TM is complex if and only if (M, D) is flat, i.e. $R \equiv 0$.[1]*

3 The Sasaki metric on TM

Definition 3.1. *Let (M, g) be a Riemannian manifold. The induced metric $\bar{g}$ on TM by g defined by*

$$\bar{g}(X, Y) = [g(d\pi(x), d\pi(Y) + g(KX, KY)]o\pi, \quad (3)$$
$$\forall X, Y \in \tau(TM);$$

is called Sasaki metric.

Since $d\pi oJ = K$ and $KoJ = d\pi$, $\bar{g}$ is Hermitian with respect to almost complex structure J.

Theorem 3.2. *Let $\bar{\nabla}$ be the Levi-Civita connection of $(TM, \bar{g})$with the Sasaki metric $\bar{g}$ and ∇ be the Levi-Civita connection of g then*

1) $(\bar{\nabla}^{Y^H}_{X^H})_{(p,u)} = (\nabla^Y_X)^H - \frac{1}{2}(R_p(X, Y)u)^V$,

2) $(\bar{\nabla}^{Y^V}_{X^H})_{(p,u)} = \frac{1}{2}(R_p(u, Y)X)^H + (\nabla^Y_X)^V_{(p,u)}$,

3) $(\bar{\nabla}^{Y^H}_{X^V})_{(p,u)} = \frac{1}{2}(R_p(u, X)Y)^H$,

4) $(\bar{\nabla}^{Y^V}_{X^V})_{(p,u)} = 0$.

4 Hessian Manifolds and their Tangent Spaces

Definition 4.1. *A Riemannian metric g on a flat manifold is called a Hessian metric if g can be locally expressed by*

$$g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}, \qquad \phi \in C^\infty(M),$$

where $\{x^1, \cdots, x^n\}$ is an affine coordinate system with respect to D. Then (M, D, g) is called a Hessian manifold.

Proposition 4.2. *Let (M, D) be a flat manifold, then (M, D, g) is a Hessian manifold if and only if*

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{kj}}{\partial x^i}, \qquad \forall i, j, k = 1, \cdots, n.$$

[7].

Definition 4.3. *Let (M, D, g) be a Hessian manifold. we denote difference tensor of D and ∇, Levi-Civita connection of g by*

$$\gamma := \nabla - D.$$

A tensor field Q of type(1,3) defined by

$$Q = D\gamma,$$

is called Hessian curvature tensor.

Proposition 4.4. *Let R be Riemannian curvature tensor for g. Then*

$$R_{ijkl} = \frac{1}{2}(Q_{ijkl} - Q_{jikl}). \quad (4)$$

[7].

4.1 The tangent bundle of Hessian manifolds

Let (M, D, g) be a Hessian manifold. With the same notation of section (1) we set

$$z^j = q^j + iq^{n+j}, \quad (5)$$

so $\{z^1, \cdots, z^n\}$ yield a holomorphic coordinate system on TM, and TM is an n-dimensional complex manifold.

Proposition 4.5. *The following conditions are equivalent.*

. *(M, D, g) is a Hessian manifold.*

. *$(TM, J, \bar{g})$ is a Kähler manifold.*

Proposition 4.6. *Every complete Hessian manifold M with constant Hessian curvature tensor has a complete Kählerian complete tangent bundle TM.*

Definition 4.7. *Let (M, D, g) be a Hessian manifold and ν be the volume element of g. we define a closed 1-form α and a symmetric 2-form β by*

$$D^\nu_X = \alpha(X)\nu,$$
$$\beta = D\alpha,$$

the α and β are called the first and the second Koszul form of g.

Proposition 4.8. *Let $R_{i\bar{j}k\bar{l}}$ and $R^T_{i\bar{j}}$ be the Riemannian curvature tensor and the Ricci tensor of Kählerian manifold $(TM, J, \bar{g})$, then we have*

$$R_{i\bar{j}} = -\frac{1}{2}\beta_{ij} o\pi, \tag{6}$$

$$R_{i\bar{j}k\bar{l}} = \frac{1}{2}Q_{ijkl} o\pi. \tag{7}$$

5 A special flow on Hessian manifolds

We suppose the Hessian manifold (M, D, g_0), then we consider an evolution equation on M by

$$\frac{\partial g_{ij}}{\partial t} = \frac{1}{2}\beta_{ij}, \tag{8}$$

where β is the second Koszul form. This equation defines a geometric flow on M and we show that the metrics along the flow remain Hessian with respect to affine coordinate system of D.

Lemma 5.1. *The evolved metrics by (8) remain Hessian along the flow.*

Proposition 5.2. *The flow (8) on Hessian manifold (M, D, g) induces Kähler Ricci flow on tangent bundle $(TM, J, \bar{g})$.*

Theorem 5.3. *Let (M, g_0) be a complete Kähler manifold with bounded nonnegative bisectional curvature. Assume that M is of maximum volume growth, that is $\nu(M, g) > 0$. Then the Kähler-Ricci , with $g(x, 0) = g_0(x)$ has a long time solution. Moreover the solution has no slowly forming singularity as t approaches ∞.[5]*

Theorem 5.4. *Let (M, g_0) be a complete Hessian manifold with constant Hessian curvature tensor $(c \geq 0)$. Then the Hessian flow (8) has a long time solution.*

Theorem 5.5. *Assume $(M, g_{i\bar{j}})$ is a complete noncompact Kähler manifold with bounded curvature and that $R_{i\bar{j}} + g_{i\bar{j}} = f_{i\bar{j}}$ for some smooth bounded function f on M. Then there exists a long time smooth solution to Kähler Ricci flow and converges, as $t \to \infty$, on every compact subset of M, to a complete Kähler- Einstein metric $\tilde{g}_{i\bar{j}}(x, \infty)$.[2]*

Corollary 5.6. *Let M be a complete Hessian manifold with constant Hessian curvature tensor and that $\beta_{ij} + g_{ij} = f_{ij}$ for some smooth bounded function f. Then Hessian flow has a long time solution and converges, as $t \to \infty$, on every compact subset of M, to a complete Hessian- Einstein metric $\tilde{g}_{ij}(x, \infty)$.*

References

[1] D., Blair. Riemannian geometry of contact and symplectic Manifolds, Second Edition, Springer Science+Business Media, LLC 2002, (2010).

[2] A., Chau. Convergence of the Kähler-Ricci flow on noncompact Kähler manifolds, J. differential geometry, 66, 211-232, (2004).

[3] S.Y., Cheng and S. T., Yau. The real monge-ampere equation and affine flat structures, Differ. Geom. and Differ. Eqns.Vol. 1, 339-370, (1982).

[4] J. L., Koszul. Domaines bornes homognes et orbites de groupes de transformation affines, Bull. Soc. Math. France, 89, 515-533, (1961).

[5] L., Ni. Ancient solutions to Kähler-Ricci flow, Math, Res, Lett. 12, 633-654, (2005).

[6] S., Sasaki. On the differential giometry of tangent bundles of Riemannian manifolds, Tohoku Math. J., 10 338-354,(1958).

[7] H., Shima. The geometry of Hessian structures, World Scientific Publ, (2007).

[8] E., Vinberg. The theory of convex homogeneous cones, Trans. Moscow Math. Soc, (1963).

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L_r-Finite Type Cones Shaped on Hyperspheres

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Abstract: We say that an isometric immersed hypersurface $x : M^n \to R^{n+1}$ is of L_r-finite type if $x = \sum_{i=0}^{p} x_i$ for some positive integer $p < \infty$, $x_i : M \to R^{n+1}$ is smooth and $L_r x_i = \lambda_i x_i$, $\lambda_i \in R$, $0 \le i \le p$, L_r is the linearized operator of the $(r+1)$th mean curvature of the hypersurface, i.e., $L_r(f) = tr(P_r \circ \nabla^2 f)$ for $f \in \mathcal{C}^\infty(M)$, where P_r is the rth Newton transformation, $\nabla^2 f$ is the Hessian of f, $L_r x = (L_r x_1, \ldots, L_r x_{n+1}), x = (x_1, \ldots, x_{n+1})$. In this paper, we prove that the only L_r-finite type cones shaped on spherical hypersurfaces are r-minimal ones.

Keywords: Cones, L_r-finite type hypersurfaces, Hessian, r-minimal, $(r+1)$th mean curvature.

1 INTRODUCTION

We say that an isometric immersed hypersurface $x : M^n \to R^{n+1}$ is of L_r-finite type if $x = \sum_{i=0}^{p} x_i$ for some positive integer $p < \infty$, $x_i : M \to R^{n+1}$ is smooth and $L_r x_i = \lambda_i x_i$, $\lambda_i \in R$, $0 \le i \le p$, L_r is the linearized operator of the $(r+1)$th mean curvature of the hypersurface, i.e., $L_r(f) = tr(P_r \circ \nabla^2 f)$ for $f \in \mathcal{C}^\infty(M)$, where P_r is the rth Newton transformation, $\nabla^2 f$ is the Hessian of f, $L_r x = (L_r x_1, \ldots, L_r x_{n+1}), x = (x_1, \ldots, x_{n+1})$. If all λ_i's are mutually different, M^n is said to be of L_r-p-type. An L_r-p-type hypersurface is said to be null if one of the $\lambda_i; 1 \le i \le p$ is zero.

The study of submanifolds of finite type began in the late seventies with B.Y. Chen's attempts to find the best possible estimate of the total mean curvature of compact submanifolds of Euclidean space and to find a notion of "degree" for submanifolds of Euclidean space (see [9] for details). Since then the subject has had a rapid development and so many mathematicians contribute of to it. The Laplace operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_r stands for the linearized operator of the first variation of the $(r+1)$th mean curvature arising from normal variations of the hypersurface (see [2]). Therefore, from this point of view, it seems natural and interesting to generalize the definition of finite type hypersurface by replacing L_r instead of Δ and study the properties of such hypersurfaces. Having this idea, for the first time, Kashani in [6] introduced such hypersurfaces and called them L_r-finite type hypersurfaces.

For a compact submanifold M of the unit hypersphere S^m of radius 1 centered at the origin, J. Simons [7] proved that if M is minimal in S^m, then $CM - \{0\}$ is minimal in R^{n+1}, where CM denotes the cone over M. In terms of finite type terminology, Simons' result says that $CM - \{0\}$ is of 1-type when M is of 1-type. In [4], O.J. GARAY generalized Simons' Theorem and showed that $CM - \{0\}$ is of finite type if and only if M is minimal in S^m. Here we generalize the result of [4] for conic hypersurface and prove the following Theorem.

Theorem: The conic hypersurface $CM - \{0\}$ is of L_r-finite type if and only if M is r-minimal in S^m.

2 Preliminaries

In this section, we recall some prerequisites about Newton transformations P_r and their associated second order differential operators L_r from [1, 3, 10, 5].

Consider an isometrically immersed hypersurface $x : M^n \to R^{n+1}$ in the Euclidean space, with the Gauss map N. We denote by ∇^0 and ∇ the Levi-Civita connections on R^{n+1} and M, respectively, then, the basic Gauss and Weingarten formulae of the hypersurface are written as

$$\nabla^0_X Y = \nabla_X Y + < SX, Y > N$$

and

$$SX = -\nabla^0_X N$$

for all tangent vector fields $X, Y \in \chi(M)$, where $S : \chi(M) \to \chi(M)$ is the shape operator (or Weingarten endomorphism) of M with respect to the Gauss map N. As is well known, S defines a self-adjoint linear operator on each tangent plane T_pM, and its eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$ are the principal curvatures of the hypersurface. Associated to the shape operator there are n algebraic invariants given by

$$s_r(p) = \sigma_r(\kappa_1(p), \ldots, \kappa_n(p)), \qquad 1 \leq r \leq n,$$

where $\sigma_r : R^n \to R$ is the elementary symmetric function in R^n given by

$$\sigma_r(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \ldots x_{i_r}.$$

Observe that the characteristic polynomial of S can be written in terms of the s_r as

$$Q_s(t) = det(tI - S) = \sum_{r=0}^{n} (-1)^r s_r t^{n-r}, \qquad (1)$$

where $s_0 = 1$ by definition. The rth mean curvature H_r of the hypersurface is then defined by

$$\binom{n}{r} H_r = s_r, \qquad 0 \leq r \leq n.$$

In particular, when $r = 1$

$$H_1 = \frac{1}{n}\sum_{i=1}^{n} \kappa_i = \frac{1}{n} tr(S) = H$$

is nothing but the mean curvature of M, which is the main extrinsic curvature of the hypersurface. On the other hand, $H_n = \kappa_1 \cdots \kappa_n$ is called the Gauss-Kronecker curvature of M. A hypersurface with zero $(r+1)$th mean curvature in R^{n+1} is called r-minimal (see [8]).

The classical Newton transformations $P_r : \chi(M) \to \chi(M)$ are defined inductively by

$$P_0 = I \text{ and } P_r = s_r I - S \circ P_{r-1} = \binom{n}{r} H_r I - S \circ P_{r-1}$$

for every $r = 1, \ldots, n$ where I denotes the identity in $\chi(M)$. Equivalently,

$$P_r = \sum_{j=0}^{r} (-1)^j s_{r-j} S^j = \sum_{j=0}^{r} (-1)^j \binom{n}{r-j} H_{r-j} S^j. \qquad (2)$$

Note that by the Cayley-Hamilton theorem stating that any operator T is annihilated by its characteristic polynomial, we have $P_n = 0$ from (1).

Each $P_r(p)$ is also a self-adjoint linear operator on the tangent space T_pM which commutes with $S(p)$. Indeed, $S(p)$ and $P_r(p)$ can be simultaneously diagonalized: if $\{e_1, \ldots, e_n\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\lambda_1(p), \ldots, \lambda_n(p)$, respectively, then they are also the eigenvectors of $P_r(p)$ with corresponding eigenvalues given by

$$\mu_{i,r}(p) = \sum_{i_1 < \cdots < i_r, i_j \neq i} \lambda_{i_1}(p) \cdots \lambda_{i_r}(p), \qquad (3)$$

for every $1 \leq i \leq n$.

Associated to each Newton transformation P_r, we consider the second-order linear differential operator $L_r : C^\infty(M) \to C^\infty(M)$ given by

$$L_r(f) = tr(P_r \circ \nabla^2 f).$$

Here, $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by

$$< \nabla^2 f(X), Y > = < \nabla_X(\nabla f), Y >, \qquad X, Y \in \chi(M).$$

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Let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame on M and observe that

$$\text{div}(P_r(\nabla f)) = \sum_{i=1}^{n} < (\nabla_{E_i} P_r)(\nabla f), E_i > + \sum_{i=1}^{n} < P_r(\nabla_{E_i} \nabla f), E_i >=< \text{div} P_r, \nabla f > + L_r(f), \tag{4}$$

where div denotes here the divergence on M.
Since $\text{div} P_r = 0$ (see [1]), as a consequence, from (4) one gets that

$$L_r(f) = \text{div}(P_r(\nabla f)). \tag{5}$$

Now we recall the definition of an L_r-finite type hypersurface from [5].

Definition 2.1. *An isometrically immersed hypersurface $x : M^n \to R^{n+1}$ is said to be of L_r-finite type if x has a finite decomposition $x = \sum_{i=0}^{m} x_i$, for some positive integer m satisfying the condition that $L_r x_i = \lambda_i x_i, \lambda_i \in R, 1 \leq i \leq m$, where $x_i : M^n \to R^{n+1}$ are smooth maps, $1 \leq i \leq m$, and x_0 is constant. If all λ_i's are mutually different, M^n is said to be of L_r-m-type. An L_r-m-type hypersurface is said to be null if one of the $\lambda_i; 1 \leq i \leq m$ is zero. The polynomial $p(t) = \prod_{i=1}^{m}(t - \lambda_i)$ is called the minimal polynomial of M for L_r.*

We should mention that, in a similar way to Proposition 1 of [3], if M is of L_r-finite type, then $p(L_r)(x - x_0) = 0$.

3 Proof of Theorem

We give the proof of Theorem, after stating lemmas 3.1 , 3.2 and 3.3.
Let M be a compact hypersurface of S^n. The cone over M, CM, is defined by the following map, $M \times [0, 1] \to R^{n+1}$; $(m, t) \mapsto tm$. Let H_s and H'_s denote the sth mean curvature vectors of $CM - \{0\}$ in R^{n+1} and that of M in S^n, respectively, and denote by ∇ and ∇' the connections of R^{n+1} and M. Let m be any point of M and choose a local orthonormal frame $\{E_i\}_{i=1}^{n-1}$ tangent to M so that $\nabla'_{E_i} E_j(m) = 0$. Let ξ be the unit vector field of CM given by $\frac{\partial}{\partial t}$. Then the integral curves of ξ are rays from the origin of R^{n+1}. By parallel translation in R^{n+1} along the rays of M, we can extend $\{E_i\}_{i=1}^{n-1}$ and ξ to an orthonormal local frame field on CM, which we also denote them by the same letters. At each time $t \in (-\varepsilon, \varepsilon)$ we have on CM a homothetic copy of M, M_t, which lies on $S^n(t)$, depending on t. Then we have

$$\nabla_{E_i} E_j(x, t) = \nabla'^t_{E_i} E_j(x, t) + \sigma^t(E_i, E_j)(x, t) - (1/t)\xi, \tag{6}$$

where ∇^t and σ^t are the Levi-Civita connection and the second fundamental form of M_t in R^{n+1}; respectively. Since M and M_t are homothetic and $\nabla'_{E_i} E_j(m) = 0$, from (6) we obtain that

$$\nabla_{E_i} E_j(m, t) = (1/t^2)\sigma(E_i, E_j)(m) - (1/t)\xi. \tag{7}$$

Lemma 3.1. *Let $x : M^{n-1} \to S^n$ be an isometric immersion of a compact hypersphere M into S^n, and $CM - \{0\}$ be the punctured cone over M. Then we have*

$$< SE_i, E_j >= (1/t^2) < S'E_i, E_j >, \; S\xi = 0, \tag{8}$$

$$H_r(m, t) = \left((n + 1 - r)/(n + 1)t^{2r}\right) H'_r(m), \tag{9}$$

$$< P_r E_i, E_j >= (1/t^{2r}) < P'_r E_i, E_j >, \; P_r \xi = \binom{n+1}{r} H_r \xi, \tag{10}$$

where S (P_r) and S' (P'_r) are the shape operators (rth Newton transformations) of $CM - \{0\}$ in R^{n+1} and that of M in S^n, respectively.

Lemma 3.2. *Under the hypothesis of Lemma 3.1, for a smooth function f on $CM - \{0\}$ we have*

$$(\overline{L}_r f)(m, t) = \frac{1}{t^{2r}}[\frac{1}{t^2}(L_r f_t)(m) + \binom{n}{r} H'_r(m) (\frac{\partial^2 f}{\partial t^2}(m, t) + \frac{(n - 2r)}{t}\frac{\partial f}{\partial t}(m, t))], \tag{11}$$

where $\overline{L}_r$ and L_r denote the linearized operators of the $(r + 1)$th mean curvature of $CM - \{0\}$ and M, respectively, and f_t is defined by $f_t(m) = f(m, t)$, $t \in (0, 1]$.

Lemma 3.3. *If $x : M^{n-1} \to S^n$ is an isometric immersion of a compact hypersphere M into S^n, then for any integer l we have*

$$\overline{L}_r^{l+1}(x,t) = ((n-r)/(n+1)t^{(2r+2)(l+1)})\{L_r^{l+1}x + \beta d_1^l L_r^l x + \beta^2 d_2^l L_r^{l-1}x + \cdots + \beta^{l-1} d_{l-1}^l L_r^2 x + \beta^l d_l^l L_r x\}, \quad (12)$$

where $\beta = \binom{n}{r}H_r'(m)$, $d_l^j = \sigma_l(a_1, \ldots, a_j)$, $a_j = (-2r-2)j((n-2r)+(-2r-2)j-1)$ and $\sigma_l(x_1, \ldots, x_j)$ denote the elementary symmetric functions in $x_1, \ldots, x_j$.

Now we suppose there exists a minimal polynomial of degree p, $Q(t) = t^p + \alpha_1 t^{p-1} + \cdots + \alpha_p$, such that

$$\overline{L}_r^{p+1}(x,t) + \alpha_1 \overline{L}_r^p(x,t) + \cdots + \alpha_p \overline{L}_r(x,t) = 0, \quad \alpha_i \in R. \quad (13)$$

Then, by substituting (11) into (13), we have

$$L_r^{p+1}x + (\beta d_1^p + \alpha_1 t^{2(r+1)})L_r^p x + (\beta^2 d_2^p + \alpha_1 \beta d_1^{p-1} t^{2(r+1)} + \alpha_2 t^{4(r+1)})L_r^{p-1}x + \cdots + (\beta^p d_p^p + \alpha_1 \beta^{p-1} d_{p-1}^{p-1} t^{2(r+1)} + \cdots + \alpha_{p-1}\beta d_1^1 t^{(2p-2)(r+1)} + \alpha_p t^{2p(r+1)})L_r x = 0, \quad (14)$$

where $\beta = \binom{n}{r}H_r'(m)$. Since equation (14) holds for every (m,t) in $CM - \{0\}$, (14) is true for any t in $(0,1]$ at every chosen point m in M. Therefore, we conclude that $H_{r+1}' = 0$. The converse follows from (9).

References

[1] L. J. Alias, N. Gürbüz, "An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures," Geom. Dedicata, vol. 121, pp. 113-127, 2006. .

[2] B. Y. Chen, "Some open problems and conjetures on submanifolds of finite type," Soochow J. Math. vol. 17, pp. 169-188, 1991.

[3] B. Y .Chen, S. Ishikawa, "On classification of some surfaces of revolution of finite type," Tsukuba J. Math. vol. 17, pp. 287-298, 1993.

[4] O. J. Garay, "Finite type cones on spherical submanifolds," proceedings of the american mathematical society, vol. 104, pp. 868-870, 1988.

[5] S. M. B. Kashani, "On some L_1-finite type (hyper)surfaces in R^{n+1}," Bull. Korean Math. Soc, vol. 46, pp. 35-43, 2009.

[6] R .C. Reilly, "Variational properties of functions of the mean curvatures for hypersurfaces in space forms", J.Differential Geom, vol. 8, pp. 465-477, 1973.

[7] J. Simons, "Minimal varieties in riemannian manifolds," Ann. of Math. vol. 88, pp. 62-105, 1968.

[8] B. G. Yang, X. M. Liu, "r-minimal hypersurfaces in space forms," J.Geom.Phys., vol. 59, pp. 685-692, 2009.

Books:

[9] B. Y. Chen, Total Mean Curvature and Submanifolds of Finite Type, Series in Pure Mathematics, 1. World Scientific Publishing Co, Singapore, 1984.

[10] J. Jost, Riemannian Geometry and Geometric Analysis, Berlin: Springer-Verlag, 1993.

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$SO(4)$ as a 2-plectic manifold

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Abstract: In this talk the Lie group $SO(4)$ is considered as a 2-plectic manifold. The standard 2-plectic structure on $SO(4)$ is induced by the Killing form. This 2-plectic structure induces a Cartan connection on $SO(4)$. The torsion and curvature tensors of this connection are calculated. Furthermore we show that there are 2-plectic structures on $\frac{SO(4)}{SO(3)} \simeq S^3$ and $\frac{SO(4)}{SO(2)}$ induced by closed left invariant 3-forms on $SO(4)$.
Keywords: 2-plectic manifold, Killing form, Cartan connection.

1 INTRODUCTION

A 2-plectic manifold is a a smooth manifold M endowed with a closed 3-form ω on it which is nondegenerate, in the sense that if $\iota_X\omega = 0$ for $X \in TM$, then $X = 0$. 2-plectic structures arise in the geometric formulation of classical field theory much in the same way that symplectic structures emerge in the Hamiltonian description of classical mechanics ([3],[2],[1]). In this formulation, a 3-dimensional field theory is described by a finite dimensional 2-plectic manifold (M, ω) as a " multi-phase space " instead of an infinite dimensional phase space. Using the 2-plectic form ω, one can define a system of partial differential equations which are the analogue of Hamiltons equations in classical mechanics.
Let G be a compact semisimple Lie group with Lie algebra g and let $\langle ., . \rangle$ denotes the positive definite adjoint invariant inner product on g defined by

$$\langle A, B\rangle = -K(A, B),$$

where K is the Killing form on g and $A, B \in g$. Assume that σ is the biinvariant Riemannian metric on G induced by $\langle ., . \rangle$. Define the 3-form ω on G by

$$\omega(X, Y, Z) = \sigma(X, [Y, Z]), \quad X, Y, Z \in TG.$$

Since σ is nondegenerate and biinvariant then so is ω. Thus ω is closed. Furthrmore, since

$$K(A, [B, C]) = K(B, [C, A]) = K(C, [A, B]),$$

for all A, B, C in g, then ω is totally antisymmetric. So ω is a 2-plectic structure on G. Choose a basis $\mathcal{E} = \{e_1, ..., e_m\}$ for g and let $E_1, ..., E_m$ are left invariant vector fields induced by $e_1, ..., e_m$, and $\Theta^1, ..., \Theta^m$ are left invariant 1-forms dual to $E_1, ..., E_m$, respectively. Denote by C^k_{ij} the structure constants of g and by $\sigma_{ij}$the components of σ

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with respect to $\mathcal{E}$. Then the 2-plectic form ω reads

$$\omega = \Sigma_{ijkl}\sigma_{ij}C^j_{kl}\Theta^i \wedge \Theta^k \wedge \Theta^l.$$

For more details about 2-plectic manifolds and Lie groups refer to [2],[5].
We recall that a connection ∇ on G is called left invariant if for any two left invariant vector fields X,Y, the vector field $\nabla_X Y$ is also left invariant. A left invariant connection is called a **Cartan connection**, if for any $A \in g$, the curve $t \mapsto e^{tA}$ is a geodesic. It is well known that there is a one-to-one correspondence between the left invariant connections on G and bilinear forms on g with value in g ([?]). Using the 2-plectic structure ω, we construct a left invariant connection on a compact semisimple Lie group G. To do this, let $\flat : g^* \to g$ be the isomorphism induced by σ. The 2-plectic form ω induces a bilinear form $\alpha : g \times g \to g$ as follows

$$\alpha(A, B) = (\iota_A \iota_B \omega_e)^\flat,$$

where e is the identity element of G. Now the left invariant connection ∇ induced by ω is defined by

$$\nabla_{E_i} E_j = (\alpha(e_i, e_j))^L,$$

where A^L denotes the left invariant vector field induced by $A \in g$. The connection ∇ is a Cartan connection, since α is skew-symmetric.

2 The 2-plectic structure on $SO(4)$

Consider the set $\mathcal{E} = \{e_1, ..., e_6\}$ as a basis for Lie algebra $so(4)$ of the Lie group $SO(4)$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Since $e_i e_j - e_j e_i = [e_i, e_j] = \sum C^k_{ij} e_k$, then the nonzero constant structures C^k_{ij} with respect to $\mathcal{E}$ are as follows

$$C^4_{12} = -1, C^5_{13} = -1, C^6_{23} = -1, C^6_{45} = -1.$$

Notice that $C^k_{ij} = -C^k_{ji} = -C^j_{ik}$. On the other hand since $K(A, B) = -tr(AB)$, one can see that $\sigma_{ij} = 2\delta_{ij}$. Thus the 2-plectic form on $SO(4)$ reads

$$\omega = -12(\Theta^1 \wedge \Theta^2 \wedge \Theta^4 + \Theta^1 \wedge \Theta^3 \wedge \Theta^5 + \Theta^2 \wedge \Theta^3 \wedge \Theta^6$$

$$+\Theta^4 \wedge \Theta^5 \wedge \Theta^6).$$

Remark 2.1. *Notice that the 3-forms $\omega_1 = d(\Theta^1 \wedge \Theta^6)$, $\omega_2 = d(\Theta^2 \wedge \Theta^5)$ and $\omega_3 = d(\Theta^3 \wedge \Theta^4)$, also are 2-plectic structures on $SO(4)$ which are not 2-plectomorphic to ω, in the sense that there is no diffeomorphism $\varphi : SO(4) \to SO(4)$ satisfying $\varphi^*\omega_i = \omega$, $i = 1, 2, 3$. Since these 2-plectic structures are exact but ω is not.*

Theorem 2.2. *Let ∇ be the Cartan connection on $SO(4)$ induced by ω, and T, R are its torsion and curvature tensors respectively. For any X, Y, Z in $T(SO(4))$ the following statements hold*
a. $\nabla_X Y = \frac{-1}{12}[X, Y]$,
b. $T(X, Y) = \frac{-7}{6}[X, Y]$,
c. $R(X, Y)Z = \frac{1}{72}[[X, Y], Z]$.

3 The $SO(4)$-Homogeneous spaces

In this section we show that some $SO(4)$-homogeneous spaces have 2-plectic structures induced by closed left invariant 3-forms on $SO(4)$.

Theorem 3.1. *The homogeneous spaces $\frac{SO(4)}{SO(2)}$ and $\frac{SO(4)}{SO(3)} \simeq S^3$ admit 2-plectic structures induced by a closed left invariant 3-form on $SO(4)$.*

Sketch of proof. Consider the closed left invariant 3-form $\alpha = \Theta^3 \wedge \Theta^5 \wedge \Theta^6$ on $SO(4)$. Then

$$Ker\alpha_e = \{v \in so(4) : \iota_v \alpha = 0\} = Span\{e_1, e_2, e_4\},$$

where e is the identity element of $SO(4)$. But

$$Span\{e_1, e_2, e_4\} = T_e(SO(3)) \subset T_e(SO(4)).$$

Thus α induces a foliation on $SO(4)$ with leaves diffeomorphic to $SO(3)$ and hence it induces a 2-plectic structure on $\frac{SO(4)}{SO(3)} \simeq S^3$. A similar argument works for $\frac{SO(4)}{SO(2)}$, when we consider the 3-form $\beta = d(\Theta^2 \wedge \Theta^4 - \Theta^3 \wedge \Theta^5)$.

Theorem 3.2. *There is no a 2-plectic structure on $\frac{SO(4)}{SO(2)\times SO(2)}$ induced by closed left invariant 3-forms on $SO(4)$.*

Sketch of proof. $SO(2) \times SO(2)$ imbeds into $SO(4)$ in two ways. Thus $T_e(SO(2) \times SO(2))$ spans by $\{e_1, e_6\}$ or $\{e_2, e_5\}$. Now let $\nu = \sum_{i<j<k} \nu_{ijk} \Theta^i \wedge \Theta^j \wedge \Theta^k$ be a closed left invariant 3-form on $SO(4)$ such that it induces a 2-plectic structure on $\frac{SO(4)}{SO(2)\times SO(2)}$. Since $\iota_{e_1}\nu = \iota_{e_6}\nu = 0$, then ν reads

$$\nu = \nu_{234}\Theta^2 \wedge \Theta^3 \wedge \Theta^4 + \nu_{235}\Theta^2 \wedge \Theta^3 \wedge \Theta^5$$
$$+\nu_{245}\Theta^2 \wedge \Theta^4 \wedge \Theta^5 + \nu_{345}\Theta^3 \wedge \Theta^4 \wedge \Theta^5.$$

Now imposing the condition $d\nu = 0$ proves the statement.

References

[1] J. C. Baez, A. E. Hoffnung, and C. L. Rogers, Categorified Symplectic Geometry and the Classical String, *Comm. Math.Phys.* **293** (2010), 701-725 .

[2] F. Cantrijn, A. Ibort, and M. DeLeon, On the Geometry of Multisymplectic Manifolds, *J. Austral. Math. Soc.(Series A),* **66** (1999),303-330 .

[3] J. F. Carinena, M. Crampin, and L. A. Ibort, On the Multisymplectic Formalism for First order Field Theories, *Diff. Geom. Appl.* **1** (1991), 345-374.

[4] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, *GSM, Vol. 34. AMS, first edition,* 2001.

[5] M. Shafiee, "On Compact Semisimple Lie Groups as 2-plectic Manifolds," to appear in J. Geom.

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Two separation axioms in Generalized Topological Spaces

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Abstract: In this paper we introduce and study μ-completely regular generalized topological spaces and $\mu T\frac{3}{4}$ separation axiom in generalized topological spaces. Some new results in generalized submaximal spaces and generalized door spaces are obtained.
Keywords: Generalized topological spaces, Submaximal GTS, μ-completely regular GTS, Generalized G_δ-Door Spaces, $\mu T\frac{3}{4}$.

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1 INTRODUCTION

Let X be a non-empty set. A subset μ of $P(X)$ is called a generalized topology (GT as an acronym) on X if it contains $\emptyset$ and any union of elements of μ belongs to μ [2]. A set X with a generalized topology μ on it, is called a generalized topological space (GTS as an acronym) and is denoted by (X, μ). A subset A of X is called μ-open (or μ-closed) if $B \in \mu$ (or $X - B \in \mu$). A GTS (X, μ) is called strong [3] if $X \in \mu$. For $A \subseteq X$, we show the union of all μ-open subsets of A, by $i_\mu(A)$ and the intersection of all μ-closed sets containing A by $c_\mu(A)$. $i_\mu(A)$ and $c_\mu(A)$ are called the interior and closure of A, respectively [2].

2 on Submaximal GTS

A point $x \in X$ is called a μ-cluster point [9] (or μ-limit point) of a set A if $U \cap (A - \{x\}) \neq \emptyset$ for each $U \in \mu$ with $x \in U$. E. Ekici [8] defined a GTS to be generalized submaximal (or a submaximal GTS) if each μ-dense subset is a μ-open set.

Let (X, μ) be a GTS and $A \subseteq X$. The μ-frontier (or μ-boundary) of A [14] is defined to be $\partial_\mu A = c_\mu(A) \cap c_\mu(A^c)$. It is known that in a submaximal topological space X, the frontier of every subset A of X has empty interior. Not all the generalized submaximal spaces have this property as it is shown in Example 2.1.

Example 2.1. *Let* $X = \{a, b, c, d\}$ *and* $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. *It is easy to check that* (X, μ) *is a submaximal GTS, but for* $A = \{a, c\}$ *we have* $i_\mu(\partial_\mu A) = \{c, d\}$.

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3 $\mu T_{\frac{3}{4}}$

Let A be a subset of a GTS (X,μ). Then A is called g_μ-closed [15] if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$. A GTS (X,μ) is called $\mu T_{\frac{1}{2}}$ [15] if every g_μ-closed subset of X is μ-closed. A GTS (X,μ) is called μT_D [9] if $\{x\}$ is an intersection of a μ-open and a μ-closed subset of X, for every $x \in X$.

Lemma 3.1. *Every srtong $\mu T_{\frac{1}{2}}$ GTS is μT_D.*

Being strong is necessary in Lemma 3.1 as it is illustrated in the following example:

Example 3.2. *Let $X = \{a,b\}$ and $\mu = \{\emptyset, \{a\}\}$. Then (X,μ) is a $\mu T_{\frac{1}{2}}$ GTS which is not μT_D.*

The converse of Lemma 3.1 is not true in general, even if (X,μ) is strong. $A \subseteq X$ is μ-regular-open (briefly, μr-open) [6] if and only if $A = i_\mu c_\mu A$, and $B \subseteq X$ is μ-regular-closed (briefly, μr-closed) if and only if $B = c_\mu i_\mu B$. A subset $\delta(\mu) = \delta$ of $P(X)$ is defined in such a way that $A \subseteq X$ belongs to δ if and only if for every $x \in A$, there exists a μ-closed set F so that $x \in i_\mu F \subseteq A$. It is well known that δ is a GT on X [6]. The elements of δ are exactly the unions of μr-open sets [6]. A set $A \subseteq X$ is δ-closed if and only if $A = c_\delta A$ [6]. A set $A \subseteq X$ is δ-open if and only if $A = i_\delta A$ [6].

Definition 3.3. *A subset A of a GTS (X,μ) is called δ-generalized closed (or g_δ-closed) if $A \subseteq U$ and $U \in \mu$ implies $c_\delta(A) \subseteq U$.*

Definition 3.4. *A GTS (X,μ) is $\mu T_{\frac{3}{4}}$ if every g_δ-closed subset of X is δ-closed.*

Theorem 3.5. *The following are equivalent for a GTS (X,μ):*

(1) *X is $\mu T_{\frac{3}{4}}$.*

(2) *For every $x \in X$, $\{x\}$ is δ-open or μ-closed.*

(3) *For every $x \in X$, $\{x\}$ is μr-open or μ-closed.*

Corollary 3.6. *Every μT_1 GTS is $\mu T_{\frac{3}{4}}$.*

The converse of Corollary 3.6 is not true in general.

Corollary 3.7. *Every $\mu T_{\frac{3}{4}}$ GTS is $\mu T_{\frac{1}{2}}$.*

The converse of Corollary 3.7 is not necessarily true.

4 μ-completely regular generalized topological spaces

Let (X,μ) and (Y,ν) be two generalized topological spaces. A function $f : (X,\mu) \to (Y,\nu)$ is said to be (μ,ν)-continuous [2] if for any ν-open set U in Y, $f^{-1}(U)$ is μ-open in X. As [9] a GTS (X,μ) is μT_2 if and only if for every $x, y \in X$, $x \neq y$, there exist $U_x, U_y \in \mu$ such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. It is well known that a finite Hausdorff topological space is discrete , but there is a non-discrete Hausdorff GTS which is finite. A GTS (X,μ) is called μT_1 [9] if for every $x, y \in X$, $x \neq y$ there exists a $U \in \mu$ such that $x \in U$ and $y \notin U$.

Definition 4.1. *Let R be the real line with the ordinary topology. If a μT_1 GTS (X,μ) has a base consisting of μ-open and μ-closed sets, then it is called μ-zero-dimensional. A μT_1 GTS (X,μ) is called μ-completely regular if for every $x \in X$ and every closed subset $F \subset X$ such that $x \notin F$ there exists a (μ, R)-continuous function $f : X \to R$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in Y$. If the cardinality of every nonempty member of μ is greater than one, then we say that (X,μ) is crowded.*

We give an example of a crowded μ-completely regular GTS which is finite.

Example 4.2. *Let $X = \{a,b,c,d\}$ and $\mu = expX \setminus \{\{a\},\{b\},\{c\},\{d\}\}$. Then (X,μ) is a zero-dimensional space since*

$$\beta = \{\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}\}$$

is a base for the GTS such that every member of β is closed. Therefore the generalized topological space is μ-completely regular.

Definition 4.3. *We call a GTS (X, μ) a generalized door space (or a μ-door space) if every subset of X is either μ-open or μ-closed.*

Example 4.4. *If $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\nu = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, then (X, μ) and (X, ν) are both generalized door spaces.*

It is easy to prove that $\{x\}$ is μ-open if and only if x is not a μ-cluster point. Contrary to topological spaces, a μT_2 generalized door space may have more than one μ-cluster point. The following example is an evidence:

Example 4.5. *Let $X = \{a, b, c, d\}$ and $\mu = P(X) - \{\{a\}, \{b\}\}$. Then (X, μ) is a μT_2 generalized door space having $\{a\}$ and $\{b\}$ as μ-cluster points.*

Theorem 4.6. *Every strong generalized door space is generalized submaximal.*

Remark 4.7. *Being strong is necessary in Theorem 4.6. Suppose that $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then (X, μ) is a generalized door space which is not generalized submaximal.*

The converse of Theorem 4.6 is not true in general, as it is shown in the following example:

Example 4.8. *Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$. Then (X, μ) is a submaximal GTS which is not a generalized door space.*

Theorem 4.9. *Every μ-subspace of a generalized door space is again a generalized door space.*

In the rest of this paper we study μ-completely regula generalized topological spaces. Finally we consider C(X), the set of all continuous functions from a GT space (X,) to the real line with the standard topology s,

References

[1] Á. Császár, Generalized open sets, *Acta. Math. Hungar.*, **75** (1997), 65-87.

[2] Á. Császár, Generalized topology, generalized continuity, *Acta. Math. Hungar.*, **96** (2002), 351-357.

[3] Á. Császár, Extremally disconnected generalized topologies, *Ann. Univ. Sci. Budapest*, **47** (2004), 91-96.

[4] Á. Császár, Separation axioms for generalized topologies, *Acta. Math. Hungar.*, **104** (2004), 63-69.

[5] Á. Császár, Generalized open sets in generalized topologies, *Acta. Math. Hungar.*, **106** (2005), 53-66.

[6] Á. Császár, δ- and θ-Modifications of generalized topologies, *Acta. Math. Hungar.*, **120** (2008), 275-279.

[7] E. Ekici, Generalized hyperconnectedness, *Acta. Math. Hungar.*, **133** (2011), 140-147.

[8] E. Ekici, Generalized submaximal spaces, *Acta. Math. Hungar.*, **134** (2012), 132-138.

[9] X. Ge and Y. Ge, μ-Separations in generalized topological spaces, *Appl. Math. J. Chinese Univ.*, **25** (2010), 243-252.

[10] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, **70** (1963), 36-41.

[11] A. S. Mashhour, M. E. Abbd El-Monsef and S. N. El Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt*, **53** (1982), 47-53.

[12] W.K. Min, Some results on generalized topological spaces and generalized systems, Acta Math. Hunger., **108** (2005), 171-181.

[13] O. Njastad, On some classes of nearly open sets, *Pacific J. Math.*, **15** (1965), 961-970.

[14] V. Renukadevi and D. Sivaraj, On δ-open sets in γ-spaces, *Filomat*, **22** (2008), 97-106.

[15] M. S. Sarsak, New separation axioms in generalized topological spaces, *Acta. Math. Hungar.*, **132** (2011), 244-252.

[16] M. S. Sarsak, On some properties of generalized open sets in generalized topological spaces, *Demonstr. Math.*, **46** (2013), 415-427.

[17] M. S. Sarsak, Weak separation axioms in generalized topological spaces, *Acta. Math. Hungar.*, **131** (2011), 110-121.

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Equivariant Morse Inequalities

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Abstract:

Classical Morse theory provides a powerful tool to obtain topological structure of a closed manifold M from the local behavior of a Morse-Smale function $f : M \to \mathbb{R}$ around its critical points. A direct result of this construction are Morse inequalities that provide lower bounds for the number of critical points of f in term of Betti numbers of M. These inequalities can be deduced through a purely analytic method by studying the asymptotic behavior of the deformed Laplacian operator. This method was introduced by E. Witten and has inspired a numbers of great achievements in Geometry and Topology in few past decades. In this paper, adopting the Witten approach, we provide an analytic proof for the so called equivariant Morse inequalities when the underlying manifold is acted on by the Lie group $G = S^1$ and the Morse function f is invariant with respect to this action.

Keywords: Morse inequalities, Equivariant Cohomology, Invariant Morse function, Deformed Laplacian.

1 INTRODUCTION

Classical Morse theory with all its variants are among the great achievement of the modern geometry and topology. Its relevance to topology begun by the fundamental observation that the cellular structure of a closed manifold M can be reconstructed through level sets of a Morse function f on M. In particular this give a way to reconstruct the cellular chain complex and therefore the cohomology of M, as is explained clearly in [?] and [?].

The seminal paper of E. Witten [?] which was inspired by ideas from quantum field theory, shed a new light on Morse theory by providing a new chain complex for computing the cohomology of M. This complex; called Morse-Smale-Witten complex; is generated by the critical points of f and graded by their Morse indices, as in the cellular complex. However its differentials are defined through the gradient lines between critical points whose indices differ by one. We refer to [?] for a detailed exposition of this theory. Amongst others, this construction led to the innovation of Floer Homology and solved (partially) the Arnold conjecture, c.f. [?].

An immediate consequence of the reconstruction of the singular cohomology via critical points of a Morse function is the Morse inequalities. Roughly speaking, they provide lower bounds for the number of critical points in term of the Betti numbers of M, i.e. the rank of the cohomology spaces of M. As far as one is interested in

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these inequalities rather than the cohomology itself, there is a very elegant and conceptual analytic derivation. This is the Witten idea of deforming the de Rham complex in an appropriate way using the Morse function and then study the asymptotic behavior of this complex.

Morse theory can be generalized in other directions. For instance in some situation there is a Lie group G acting on M and preserves f. This problem naturally arises in n-body problem where the central configurations are critical points of some Morse function which are symmetric with respect to the action of $SO(n)$ (c.f. [?]). Another example is the problem of finding the number of closed geodesics of a Riemannian metric where the Lie group is S^1. Actually this last example was among the first applications of Morse theory that was worked out by Morse in [?], see also [?, chapter 3]. In these cases the critical levels of f are clearly orbits of the action. The Morse-Bott theory ignore the invariance of f under the action and provides lower-bounds for the number of these critical levels (see [?] and [?, page 344]). However; as it is clearly explained in [?, pages 351-355]; to get best results one has to assume the G-invariance of f and this requires an appropriate cohomology theory that takes the group action into its construction.

This is the *equivariant cohomology theory* which is introduced originally by E. Cartan in 1940s. As in ordinary Morse-Bott theory, the equivariant cohomology may be reconstructed from a cellular chain complex generated by critical level of f. This is done by A. G. Wasserman in [?, section 4] for a compact Lie group G. Nevertheless the Morse-Smale-Witten complex for equivariant cohomology has been worked out recently, for $G = S^1$, by M. J. Berghoff in his PhD thesis [?]. This constructions lead to the equivariant Morse inequalities whose precise statement might be found in [?, page 149] or in [?, page 351].

In this paper we consider also the case $G = S^1$ and prove the equivariant Morse inequalities by adopting the Witten method to deform the Borel complex.

2 Preliminaries and Results

Let $G = S^1$ be a compact Lie group that acts on a topological space X and let EG be a contractible space that is acted on, freely and continuously, by G. Such spaces exist and are unique up to homotopy equivalence and the quotient $BG := EG/G$ is the classifying space of G. The diagonal action of G on $X \times EG$ is free and the quotient $X_G := (X \times EG)/G$ is called the homotopy path space. The equivariant cohomology of X; that we denote by $H^*_G(X)$; is by definition the singular cohomology (with coefficient in $\mathbb{C}$) of the homotopy path space

$$H^*_G(X) = H^*(X_G) \tag{1}$$

When X is a differential manifold (that we denote by M from now on) and the action is smooth there is a more geometric approach to the construction of the equivariant cohomology. We consider the simplest case when the Lie group G is just the circle S^1.

Let M be a closed smooth manifold of dimension n which is acted on by the group $G = S^1$. This action is supposed to be smooth and not necessarily free. The Lie algebra of S^1 is $\mathbb{R}$ with a fixed element 1. This element generates a vector field v over M which is tangent to the orbits and vanishes at fixed points of the action. The vector field v is called the infinitesimal generator of the action and we denote its t-time flow by ϕ_t . Let $\Omega^*_G(M) \subset \Omega^*(M)$ consists of all invariant differential forms ω satisfying $\phi^*_t(\omega) = \omega$ for $t \in \mathbb{R}$, or equivalently $\mathcal{L}_v(\omega) = 0$. Consider the algebra $\Omega^*_{eq}(M) := \mathbb{C}[t] \otimes \Omega^*_G(M)$. This algebra is graded by the rule $\deg(t^k \otimes \omega) = 2k + \deg(\omega)$. The linear map

$$\begin{aligned} d_{eq} &: \Omega^*_{eq}(M) \to \Omega^*_{eq}(M) \\ d_{eq}(t^k \otimes \omega) &= t^k \otimes d\omega + t^{k+1} \otimes i_v\omega \end{aligned} \tag{2}$$

is a differential, i.e. $d_{eq}^2 = 0$ and increases the degree by one. The *equivariant de Rham cohomology groups* $H_G^*(M)$ are the cohomology groups of this graded differential complex. It turns out that these groups are isomorphic to the groups introduced by (**??**) when X is a smooth manifold.

The cohomology space $H_G^k(M)$ is a complex finite dimensional vector space. So one can define the equivariant Betti numbers by $\beta_G^k := \dim H_G^k(M)$.

Let $f : M \to \mathbb{R}$ be an invariant smooth function, i.e. $v.f = 0$. An orbit o is critical if one point on it (then all points) is critical point for f. For $x \in o$ let N_x stands for the quotient space T_xM/T_xo. For $x \in M$ and $X, Y \in T_xM$ the Hessian of f is a symmetric bi-linear form defined as follows

$$H_f(X,Y) = X.(Y.f) - (\nabla_X Y).f$$

Here ∇ is the Riemannian connection on TM associated to a Riemannian metric g. Because S^1 is compact it is always possible, throgh an averaging procedure, to assume g be S^1-invariant. With this assumption it is true that if X or Y belong to T_xo then $H_f(X,Y) = 0$. Therefore the Hessian defines a well defined symmetric bi-linear form on N. Using the Riemannian metric, we can identify N_x with the orthogonal compliment of T_xo. We denote the restriction of H to $N \subset TM$ by $\bar{H}_f$. We say a critical orbit o be transversally non-degenerated (or simply non-degenerated) if $\bar{H}_f$ is non-degenerated at any points of o. The Morse index of such orbit is the dimension of the maximal subspace of N_x, on which the Hessian is negative-definite. In the sequel we reserve the notation o for a non-trivial orbit and we denote a trivial orbit by its geometric image, that is a point p in M. Let c_k and d_k denote respectively the number of critical points and orbits with Morse index k. Our aim is to provide an analytic proof for the following equivariant Morse inequalities via Witten deformation:

Theorem 2.1. *With* $\tilde{c}_k := d_k + c_k + c_{k-2} + c_{k-4} + \ldots$ *the following inequalities hold for* $k = 0, 1, 2, \ldots$

$$\tilde{c}_k - \tilde{c}_{k-1} + \cdots \pm \tilde{c}_0 \geq \beta_{eq}^k - \beta_{eq}^{k-1} + \cdots \pm \beta_{eq}^0,$$

Actually the inequalities for $k \geq n+1$ *are the same that the inequality for* $k = n$.

Note that, when the action is free these inequalities reduce to the ordinary Morse inequalities for the function $\tilde{f} \colon \tilde{M} \to \mathbb{R}$, while it reduces to the Morse inequalities for the function f when the action is trivial.

References

[1] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[2] M. Berghoff. Sˆ1-equivariant Morse cohomology. *ArXiv e-prints*, April 2012.

[3] Raoul Bott. Nondegenerate critical manifolds. *Ann. of Math. (2)*, 60:248–261, 1954.

[4] Raoul Bott. Lectures on Morse theory, old and new. *Bull. Amer. Math. Soc. (N.S.)*, 7(2):331–358, 1982.

[5] Raoul Bott. Morse theory indomitable. *Inst. Hautes Études Sci. Publ. Math.*, (68):99–114 (1989), 1988.

[6] J. Milnor. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[7] Marston Morse. *The calculus of variations in the large*, volume 18 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 1996. Reprint of the 1932 original.

[8] Filomena Pacella. Central configurations of the N-body problem via equivariant Morse theory. *Arch. Rational Mech. Anal.*, 97(1):59–74, 1987.

[9] Steven Rosenberg. *The Laplacian on a Riemannian manifold*, volume 31 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. An introduction to analysis on manifolds.

[10] Dietmar Salamon. Lectures on Floer homology. In *Symplectic geometry and topology (Park City, UT, 1997)*, volume 7 of *IAS/Park City Math. Ser.*, pages 143–229. Amer. Math. Soc., Providence, RI, 1999.

[11] Matthias Schwarz. *Morse homology*, volume 111 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1993.

[12] Arthur G. Wasserman. Equivariant differential topology. *Topology*, 8:127–150, 1969.

[13] Edward Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692 (1983), 1982.

[14] Mostafa Esfahani Zadeh. Morse inequalities for manifolds with boundary. *J. Korean Math. Soc.*, 47(1):123–134, 2010.

[15] Mostafa Esfahani Zadeh. Delocalized Betti numbers and Morse type inequalities. *Rocky Mountain J. Math.*, 41(4):1361–1374, 2011.

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حل هندسی مساله سینماتیک معکوس یک ربات سریالی با استفاده از فرمول ضرب نگاشت نمایی

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شهریور۱۳۹۳

چکیده: دراین مقاله؛ یک روش هندسی برای حل تحلیلی سینماتیک معکوس ربات سریالی ارائه شده، دراین روش از فرمول ضرب نگاشت نمایی که به اختصارPOE می نامیم استفاده می کنیم، این نگاشت نمایی روی جبر گروه لی$SE(3)$ تعریف می شود. در این روش مساله سینماتیک معکوس به چند زیر مساله متعارف تجزیه شده که هر کدام ازآن ها قادرند مساله سینماتیک معکوس را به زیر بخش های مجزا تبدیل و آن را حل کنند.

کلمات کلیدی:
مفصل، سینماتیک مستقیم، سینماتیک معکوس، تاب، POE

مقدمه

همگام با صنعتی شدن زندگی بشر نیاز به استفاده از ابزاری که بتواند کارهای تکراری و پیچیده را انجام دهد افزایش یافت. این نیاز موجب بوجود آمدن علم رباتیک شد. در رشته رباتیک از علوم زیادی از جمله برق، مکانیک، فیزیک،ریاضی و ...استفاده می‌شود. شاخه‌های مهم علم رباتیک که از علم ریاضی استفاده می‌کنند عبارتنداز کنترل ربات، طراحی حرکت ربات، سینماتیک مستقیم ومعکوس که در این مقاله به مطالعه سینماتیک معکوس پرداخته شده‌است. در سینماتیک معکوس با داشتن موقعیت اولیه و نهایی ربات قصد داریم چگونگی اتصال مفصل‌ها، نوع حرکت، اندازه حرکت ربات را تا رسیدن به موقعیت نهایی توصیف کنیم. یکی از روش‌های معمول برای حل تحلیلی مساله سینماتیک معکوس براساس روش پارامتری‌سازی دناویت-هارتنبرگ است که به اختصار $(D-H)$ نمایش می‌دهیم. در روش$D-H$ که یک روش جبری است از مفاهیم هندسی استفاده نمیشود، روش ارائه شده در این مقاله که به اختصار POE نامیده می‌شود روشی هندسی است که بر اساس نوع هندسه قرار گرفتن مفصل‌ها نسبت به هم تعریف می‌شود. برای حل مساله سینماتیک معکوس با این روش یازده زیر مساله بیان می شود.

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پیشنیازهای هندسی

در این بخش پیش نیازهای هندسی مورد نیاز برای فرمول POE بیان می‌شود. یک جسم صلب عبارت است از یک جسم B به طوری که حرکت این جسم در فضا کاملا توسط حرکت یک نقطه دلخواه $q \in B$ و حرکت یک دستگاه محتصات متعامد اولیه $\{v, u, w\}$ که مبدا آن نقطه P می‌باشد، مشخص می‌شود و حرکت صلب حرکتی است که در طول حرکت فاصله بین نقاط و زاویه بین بردارها حفظ می‌شود.

گروه خاص اقلیدسی $SE(3)$ از حرکت جسم صلب را به صورت زیر

$$SE(3) = \left\{ g \mid g = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}, R \in SO(3), P \in R^3 \right\}$$

و جبرلی $SE(3)$ را به صورت زیر تعریف می‌کنیم:

$$se(3) = \left\{ \hat{\xi} \mid \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}, \hat{\omega} \in so(3), v, \omega \in R^3 \right\}$$

هر $\hat{\xi} \in se(3)$ را یک تاب می‌نامیم. نگاشت نمایی را روی جبرلی $SE(3)$ به صورت زیر تعریف می‌کنیم: $\exp : se(3) \longrightarrow SE(3)$ که $\hat{\xi} \longmapsto e^{\hat{\xi}}$.

سینماتیک مستقیم

در این بخش نگاشت سینماتیک مستقیم را برای یک ربات با $n-$ درجه آزادی بدست می‌آوریم.

یک ربات با $n-$ درجه آزادی را به اختصار با $n - DOF$ نشان می‌دهیم، برای توصیف هندسی سینماتیک مستقیم فرض کنید M یک جسم صلب، A دستگاه مختصات استاندارد و ثابت R^3 با مبدا $O = (0, 0, 0)$ و B دستگاه مختصات متصل به جسم M با مبدا O باشد. در این صورت اگر جسم صلب طوری حرکت کند که نقطه O ثابت بماند و نقطه $q \in M$، ثابت می‌شود که $q_a = R_{ab}q_b$ که q_a مختصات نقطه q نسبت به دستگاه مختصات A باشد و q_b مختصات نقطه q نسبت به دستگاه مختصات B باشد و $R_{ab} = (\tilde{R})^{-1}$ که $\tilde{R} = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$، (u, v, w) مختصات دستگاه B و $\tilde{R}$ ماتریس تبدیل مختصات B نسبت به دستگاه A است. حال اگر جسم صلب M حرکت کند، دستگاه B هم با آن حرکت می‌کند پس مختصات هر نقطه $q \in M$ در هر لحظه t نسبت به دستگاه مختصات A برابر با $q_a(t) = R_{ab}(t)q_b(t)$ به ماتریس $R_{ab}(t)$ ماتریس پیکربندی گوییم و به مجموعه تمام ماتریس‌های پیکربندی، فضای پیکربندی گوییم.

اگر p_{ab} را مختصات مبدا دستگاه B نسبت به A فرض کنیم خواهیم داشت $q_a = p_{ab} + R_{ab}q_b$ که R_{ab} ماتریس دوران وعضو گروه لی $SO(3)$ است، قرار می‌دهیم $g_{ab} = (R_{ab}, p_{ab}) \in SE(3) = SO(3) \times R^3$. در ادامه، حرکت مفاصل را به طور منحصر به فرد متناظر با یک تاب در نظر می‌گیریم، یعنی تاب‌ها متناظر محور مفصل‌ها هستند. هرتاب یک بردار 6×1 است که با ξ نمایش می‌دهیم. ثابت می‌شود نگاشت سینماتیک متناظر این تاب به صورت $g_{ab}(\theta) = e^{\hat{\xi}\theta}g_{ab}(0)$ است که $g_{ab}(\theta) \in SE(3)$ موقعیت نهایی ربات بعد از تبدیل حول ξ به اندازه θ می‌باشد. مفاصل ربات‌های مورد بحث در این مقاله شامل مفصل دورانی و انتقالی می‌باشند. برای مفاصل انتقالی هر ξ_i بصورت $\xi_i = \begin{bmatrix} 0 \\ v_i \end{bmatrix}$ که $v_i \in R^{3\times1}$ یک بردار یکه در جهت انتقال است.

و برای مفاصل دورانی هر ξ_i بصورت $\xi_i = \begin{bmatrix} \omega_i \\ \omega_i \times q_i \end{bmatrix}$ است که $\omega_i \in R^{3\times1}$ یک بردار یکه درجهت محور تاب و $q_i \in R^{3\times1}$ یک نقطه دلخواه روی محور تاب است.

ثابت می‌شود برای یک ربات با $n-$ درجه آزادی سینماتیک مستقیم آن به صورت زیر است:

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2}\theta_2 \dots e^{\hat{\xi}_n}\theta_n g_{st}(0) \qquad (۱)$$

سینماتیک معکوس

در سینماتیک معکوس با معلوم بودن$g_{ab}(0)$ و $g_{ab}(\theta)$ هدف یافتن مقادیر $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ است. برای اینکار باتکنیک‌هایی هندسی یکسری زیر مساله تعریف می‌کنیم وبا استفاده ازاین زیر مساله‌ها، مساله سینماتیک معکوس حل می‌شود.

نتایج

ارائه یک روش تحلیلی برای حل مساله سینماتیک معکوس ربات‌ها. مزیت این روش نسبت به دیگر روش ها استفاده از تکنیک‌های هندسی است. با روش ارائه شده ربات‌های با ۴ درجه آزادی بطور کامل، ربات‌های با ۵ درجه آزادی تا ۹۰درصد و ربات‌های با ۶ درجه آزادی تا ۵۰درصد قابل حل هستند.

مراجع

[۱] I. Chen، Y. Gao، Closed-form inverse kinematics solver for reconfigurable robots، in: International Conference on Robotics and Automation، ۲۰۰۱، pp. ۲۳۹۵—۲۴۰۰.

[۲] R.M. Murray، Z.X. Li، S.S. Sastry، A Mathematical Introduction to Robotic Manipulation، CRC Press، Boca Raton، FL، ۱۹۹۴

[۳] J.M. Selig، Geometric Methods in Robotics، second ed. Springer، New York، ۲۰۰۵.

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A Generalization of Bonnet-Myers Theorem

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Abstract: Let (M, F) be a forward geodesically complete Finsler manifold satisfying $2Ric_{jk} + \mathcal{L}_{\hat{V}} g_{jk} \geq 2\lambda g_{jk}$, where $\hat{V}$ is complete lift of the vector field V on M and $\lambda > 0$. It is shown that M is compact if and only if the norm $\|V\|$ is bounded on SM. In particular, a forward complete shrinking Finslerian Ricci soliton (M, F, V) is compact if and only if $\|V\|$ is bounded on SM, in such case, the fundamental group is finite and hence the first de Rham cohomology group is zero.

Keywords: Finsler geometry, Ricci soliton.

1 INTRODUCTION

The Ricci flow in Riemannian geometry was introduced by R. S. Hamilton in 1982, cf. [4], and since then has been extensively studied thanks to its applications in geometry, physics and different branches of real world problems. Theoretical physicists have also been looking into the equation of quasi-Einstein metrics in relation with string theory. Ricci flow is a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat. G. Perelman used Ricci flow to prove the Poincaré conjecture. Quasi-Einstein metrics or Ricci solitons are considered as a solution to the Ricci flow equation and are subject of great interest in geometry and physics. Let (M, g) be a Riemannian manifold, a triple (M, g, X) is said to be a *quasi-Einstein metric* or *Ricci soliton* if g satisfies the equation

$$2Rc + \mathcal{L}_X g = 2\lambda g, \tag{1}$$

where Rc is the Ricci tensor, X a smooth vector field on M, $\mathcal{L}_X$ the Lie derivative along X and λ a real constant. A Ricci soliton is said to be shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. If the vector field X is gradient of a potential function f, then (M, g, X) is said to be *gradient* and (1) takes the familiar form

$$Rc + \nabla\nabla f = \lambda g. \tag{2}$$

Perelman has proved that on a compact Riemannian manifold every Ricci soliton is gradient, cf. [5]. Moreover, on a compact Riemannian manifold a quasi-Einstein metric is a special solution to the Ricci flow equation defined by

$$\frac{\partial}{\partial t} g(t) = -2Rc, \quad g(t=0) := g_0. \tag{3}$$

A quasi-Einstein metric is considered as special solution to the Ricci flow in Riemannian geometry.

The concept of Ricci flow on Finsler manifolds is defined first by D. Bao, cf., [2], choosing the Ricci tensor defined by H. Akbar-Zadeh. This

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choice of D. Bao for definition of Ricci tensor is completely suitable for definition of Ricci flow in Finsler geometry. In fact, H. Akbar-Zadeh has used Einstein-Hilbert's functional in general relativity and introduced definition of Einstein-Finsler spaces as critical points of this functional, similar to Hamilton's work.

Some recent work has focused on the natural question of extending this notion to the Finsler geometry as a natural generalization of Riemannian geometry. In [3], it is proved that if there is a quasi-Einstein Finsler metric on a compact manifold then there exists a solution to the Ricci flow equation and conversely, certain form of solutions to the Ricci flow are quasi-Einstein Finsler metrics. Since Ricci solitons generalize Eienstein manifolds, it is natural to ask whether classical results like Bonnet-Myers theorem for Eienstein manifolds of positive Ricci scalar remain valid in the Ricci soliton case. In this work, it is shown that a forward geodesically complete connected Finsler manifold satisfying in certain form is compact if and only if it is bounded. More intuitively, it is proved that a compact shrinking Ricci soliton has finite fundamental group.

2 PRELIMINARIES

Let M be a real n-dimensional differentiable manifold. We denote by TM its tangent bundle and by $\pi : TM_0 \longrightarrow M$, fiber bundle of non zero tangent vectors. A *Finsler structure* on M is a function $F : TM \longrightarrow [0, \infty)$, with the following properties:
I. Regularity: F is C^∞ on the entire slit tangent bundle $TM_0 = TM\backslash 0$.
II. Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.
III. Strong convexity: The $n \times n$ Hessian matrix $g_{ij} = ([\frac{1}{2}F^2]_{y^i y^j})$ is positive definite at every point of TM_0. A *Finsler manifold* (M, F) is a pair consisting of a differentiable manifold M and a Finsler structure F. The formal Christoffel symbols of second kind and spray coefficients are respectively denoted here by

$$\gamma^i_{jk} := g^{is}\frac{1}{2}(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j}), \tag{4}$$

where $g_{ij}(x, y) = [\frac{1}{2}F^2]_{y^i y^j}$, and

$$G^i := \frac{1}{2}\gamma^i_{jk}y^j y^k. \tag{5}$$

We consider also the *reduced curvature tensor* R^i_k which is expressed entirely in terms of the x and y derivatives of spray coefficients G^i.

$$R^i_k := \frac{1}{F^2}(2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}). \tag{6}$$

In the general Finslerian setting, one of the remarkable definitions of Ricci tensors is introduced by H. Akbar-Zadeh [1] as follows.

$$Ric_{jk} : = [\frac{1}{2}F^2 Ric]_{y^j y^k},$$

where $Ric = R^i_i$ and R^i_k is defined by (6). Akbar-Zadeh's definition of Einstein-Finsler space related to this Ricci tensor is obtained as critical point of Einstein-Hilbert functional and hence suitable for definition of Finslerian Ricci flow. One of the advantages of the Ricci quantity defined here is its independence to the choice of the Cartan, Berwald or Chern(Rund) connections. Based on the Akbar-Zadeh's Ricci tensor, in analogy with the equation (3) , D. Bao has considered, the following natural extension of *Ricci flow* in Finsler geometry, cf., [2],

$$\frac{\partial}{\partial t}g_{jk} = -2Ric_{jk}, \quad g(t=0) := g_0.$$

This equation is equivalent to the following differential equation

$$\frac{\partial}{\partial t}(\log F(t)) = -Ric, \quad F(t=0) := F_0, \tag{7}$$

where, F_0 is the initial Finsler structure.

3 MAIN RESULT

Let (M, F_0) be a Finsler manifold and $V = v^i(x)\frac{\partial}{\partial x^i}$ a vector field on M. We call the triple

(M, F_0, V) a Finslerian *quasi-Einstein* or a *Ricci soliton* if g_{jk} the Hessian related to the Finsler structure F_0 satisfies

$$2Ric_{jk} + \mathcal{L}_{\hat{V}} g_{jk} = 2\lambda g_{jk}, \quad (8)$$

where, $\hat{V}$ is complete lift of V and λ is a real number. A Ricci soliton is said to be shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. It said to be a forward complete (resp. compact) Ricci soliton if (M, F) is forward complete (resp. compact). Note that according to the Hopf-Rinow theorem, two notions forward complete and forward geodesically complete are equivalent.

Theorem 3.1. *Let (M, F) be a forward geodesically complete Finsler manifold satisfying*

$$2Ric_{jk} + \mathcal{L}_{\hat{V}} g_{jk} \geq 2\lambda g_{jk}, \quad (9)$$

where, $\hat{V}$ is complete lift of V and $\lambda > 0$. Then, M is compact if and only if $\|V\| := g(V, V)^{\frac{1}{2}}$ is bounded on SM and moreover, in such case, $diam(M)$ is bounded.

Corollary 3.2. *Let (M, F, V) be a forward complete shrinking Ricci soliton. Then, M is compact if and only if $\|V\|$ is bounded on SM and moreover, in such case, $diam(M)$ is bounded.*

Theorem 3.3. *Let (M, F) be a compact Finsler manifold satisfying (9) where, $\hat{V}$ is complete lift of V and $\lambda > 0$. Then the fundamental group $\pi_1(M)$ of M is finite and therefore $H^1_{dR}(M) = 0$.*

Corollary 3.4. *Let (M, F, V) be a compact shrinking Ricci soliton. Then the fundamental group $\pi_1(M)$ of M is finite and therfore $H^1_{dR}(M) =$* 0.

References

[1] H. Akbar-Zadeh, "Sur les espaces de Finsler a courbures sectionnelles constantes," Acad. Roy. Belg. Bull. Cl. Sci. (5) **74** (1988), 199-202.

[2] D. Bao, "On two curvature-driven problems in Riemann-Finsler geometry," Adv. stud. pure Math. **48** (2007), 19-71.

[3] B. Bidabad and M. Yarahmadi, *On quasi-Eienstein Finler spaces*, Accepted in Bulletin of the Iranian Mathematical Society, to appear, 2014.

[4] R.S. Hamilton, "Three-manifolds with positive Ricci curvature," J. Differential Geom. **17** (1982), no. 2, 255-306.

[5] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications," http://arxiv.org/abs/math.DG/02111159.

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On the Geometry of Focal Schemes arising from General Prym-Canonical Curves

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Abstract: In this note we study focal schemes arising from singularities of the theta divisor of a general Prym-Canonical curve. Particularly we show that these are singular schemes.

Keywords: Focal Variety, Projectivized Tangent Cone, Prym-Canonical Curve.

1 INTRODUCTION

Throughout this lecture the field of complex numbers, C, would be considered as our base field. All the schemes and varieties are considered on this field. This manuscript of our lecture is organized in three sections. After gathering together definitions and backgrounds in section 2, we explain how one can fit the linear spaces inside projectivized Prym tangent cones of the Theta divisor of a general Prym-Canonical curve into a family, family (1). Finally in the last section, section 3, we show that the focal schemes of this family are singular schemes.

2 (g-4)-dimensional Family of (g-5)-Planes arising from Singularities of Prym-Theta Divisor

Let C be a general smooth curve of genus g and $f:\widetilde{C}\to C$, an $\acute{e}$tale double cover of C. Then by Hurwitz formula $\widetilde{C}$ is a curve of genus $2g-1$. Throughout we assume that $g\geq 7$. Denote by η the line bundle corresponding to this double cover. The line bundle η satisfies $\eta^2=0$. Denote by $J(\tilde{C})$ the jacobian of $\tilde{C}$ and let $P_f := \mathrm{Prym}(\widetilde{C},C)\subset J(\widetilde{C})$ be the prym variety associated to the double covering f, with Ξ the theta divisor of P_f. By generality of C, the divisor Ξ has no singularity of exceptional type. Let Sing(Ξ) denote the singular locus of the theta divisor Ξ. For $g=7$ the scheme Sing(Ξ) is a zero-dimensional scheme while for $g\geq 8$ it is $(g-7)$-dimensional. Let $\widetilde{\alpha}^3_{2g-2}:\widetilde{C}^3_{2g-2}\to\widetilde{W}^3_{2g-2}$ be the Abel-Jacobi map on $\tilde{C}$ and set $S=(\widetilde{\alpha}^3_{2g-2})^{-1}(\mathrm{Sing}(\Xi))$. For general curves C and for $g\geq 7$ the scheme S is of dimension $g-4$.

By geometric Riemann-Roth theorem, for any divisor $\widetilde{D}_s\in S$ the linear span of $\widetilde{D}_s$, which we denote it by $<\widetilde{D}_s>$, is a $(2g-6)$-dimensional linear subspace of $\mathbf{P}^{2g-2}:=\mathbf{P}(H^0(\widetilde{C},\omega_{\widetilde{C}})^*)$. These linear spaces fit into a $(g-4)$-dimensional family of $(2g-6)$-dimensional linear subspaces of $\mathbf{P}^{2g-2}$. We denote this family by $\underline{\widetilde{\Lambda}}$. See [1], [3], [4], [5] and [6] for simi-

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lar families and their applications.

For a line bundle $L \in \mathrm{Sing}(\Xi)$, the variety $\tilde{Q}_L$:

$$\tilde{Q}_L := \bigcup_{\tilde{D}_s \in \tilde{\alpha}^{-1}(L)} < \tilde{D}_s > \quad \subset \quad \mathbf{P}^{2g-2}$$

is by Kempf-Riemann theorem a quartic hypersurface in $\mathbf{P}^{2g-2}$. In fact $\tilde{Q}_L$ is the projectivized tangent cone of $\tilde{\Theta}$ at the point L, for which $\tilde{\Theta}$ is the theta divisor of $J(\tilde{C})$.

Consider that the space of canonical differential forms on $\tilde{C}$ splits into direct sum of the space of canonical and the space of anti-canonical differential forms on C. Therefore one has the equality

$$H^0(\tilde{C}, \omega_{\tilde{C}})^* = H^0(C, \omega_C)^* \oplus H^0(C, \omega_C \otimes \eta)^*.$$

This equality gives an inclusion:

$$\mathbf{P}(H^0(C, \omega_C \otimes \eta)^*) \subset \mathbf{P}(H^0(\tilde{C}, \omega_{\tilde{C}})^*) = \mathbf{P}^{2g-2}.$$

The generality assumption of C implies that $\tilde{Q}_L$ does not contain $\mathbf{P}^{g-2} := \mathbf{P}(H^0(C, \omega_C \otimes \eta)^*)$. Therefore

$$Q_L := \frac{1}{2}\tilde{Q}_L \cdot \mathbf{P}^{g-2}$$

is a quadric in $\mathbf{P}^{g-2}$. Moreover Q_L is a quadric at most of rank six which is singular for $g \geq 8$.

The geometry of Q_L's, when they are at most of rank 4, have been demonstrated in [2].

Now results of [8] together with generality of C imply that the variety Q_L is the projectivized tangent cone of Ξ at L. Again generality of C together with corollaries 2.7 and 2.21 of [8] imply that the linear space $\Lambda_s =< \tilde{D}_s > \cdot \mathbf{P}^{g-2}$ is of dimension $g-5$. This implies in its own right that the 3-dimensional family of $(2g-6)$-dimensional linear subspaces of $\mathbf{P}^{2g-2}$ parameterized by $\tilde{\alpha}^{-1}(L)$, cut on $\mathbf{P}^{g-2}$ the 3-dimensional family of $(g-5)$-dimensional linear subspaces Λ_s and these Λ_s's are the linear subspaces contained in projectivized tangent cone of Ξ at L.

The discussion just has been done shows that when L varies in $\mathrm{Sing}(\Xi)$, the linear subspaces contained in Q_L's fit into a family:

$$\begin{array}{ccc} \underline{\Lambda} & \subset & S \times \mathbf{P}^{g-2} \\ \downarrow & \pi & \\ S & & \end{array} \qquad (1)$$

which is a $(g-4)$-dimensional family of $(g-5)$-dimensional linear subspaces of $\mathbf{P}^{g-2}$. The focal map of this family at a general point $s \in S$

$$\phi_s : T_{S,s} \otimes \mathcal{O}_{\Lambda_s} \to \mathcal{N}_{\Lambda_s/\mathbf{P}^{g-2}}$$

is a morphism between vector bundles of ranks $g-4$ and 3 respectively. Consider the map:

$$\begin{array}{cccc} \psi : S & \to & \mathrm{Grass}(g-5, g-2) \\ s & \mapsto & \Lambda_s \end{array} \qquad (2)$$

and notice that this is a finite to one map which implies that for a general $s \in S$ the map

$$H^0(\phi_s) : T_{s,S} \to H^0(\mathcal{N}_{\Lambda_s/P^{g-2}})$$

is injective (See [3], page 5).

The injectivity of $H^0(\phi_s)$ implies that for general $s \in S$ the map ϕ_s is not identically zero. Since otherwise the map $H^0(\phi_s)$ would be a zero map which is a contradiction by it's injectivity. The discussion now concludes that the degeneracy locus of the focal map, F_s, is not the whole space of $\mathbf{P}^{g-2}$ and therefore it is expected to be a curve.

3 Non-Smoothness of Focal locus's of the family (1).

Denote by C_η the prym-canonical model of C in $\mathbf{P}^{g-2}$. Since Ξ does not have an exceptional singularity, for any line bundle $L \in \mathrm{Sing}(\Xi)$ one has the inclusion $C_\eta \subset Q_L$. For a line bundle $L \in \mathrm{Sing}(\Xi)$, set $F_L = \overline{\cup_{s \in \alpha^{-1}(L)} F_s}$.

The following Lemma is well known in the theory of Algebraic Curves.

Lemma 3.1. *For a divisor D on a smooth projective curve C, the linear series $\mid D \mid$ is of dimension $\geq r$ if and only if for any effective divisor D_1 of degree r there exists $\widetilde{D} \in \mid D \mid$ such that $\widetilde{D} \geq D_1$.*

Theorem 3.2. *For any line bundle $L \in \mathrm{Sing}(\Xi)$, the inclusion $C_\eta \subset F_L$ is valid.*

Proof. For a line bundle $L \in \mathrm{Sing}(\Xi)$, since C_η is contained in Q_L, it is enough to prove that for a linear space Λ_s contained in Q_L, one has $C_\eta \cap \Lambda_s \subset F_s$. For a point $\overline{p} \in C_\eta \cap \Lambda_s$, set $f^{-1}(\overline{p}) = \{p, q\}$. By Lemma (3.1), for a general point $t \in \tilde{C}$ there exists at least a divisor $D \in \mid \tilde{D}_s \mid$ containing the points p, q and t in its support. Therefore there is a curve $X \subset \tilde{\alpha}^{-1}(L)$ such that for any $x \in X$ the linear space $\tilde{\Lambda}_x$ generated by the divisor x, contains p and q. Therefore for any $x \in X$ one has the inclusion:

$$L_{p,q} \cap \mathbf{P}^{g-2} \subset \tilde{\Lambda}_x \cap \mathbf{P}^{g-2} = \Lambda_x$$

for which $L_{p,q}$ is the line in $\mathbf{P}^{2g-2}$ passing through the points p and q. Now [8] implies that $\overline{p} \in \Lambda_x$. Therefore there are 1-dimensional family of linear spaces Λ_x passing through $\overline{p}$. By moving the line bundle L within the scheme $\mathrm{Sing}(\Xi)$ one gets a $(g-6)$- dimensional family of Λ_s's passing through $\overline{p}$. Since $g \geq 7$, this proves that $\overline{p}$ belongs to F_s. Therefore any point $\overline{p}$ of C_η belongs to F_L. This finishes the proof. □

The proof of Theorem (3.2) implies that through any point $\overline{p}$ of C_η, it passes at least $(g-6)$-dimensional family of Λ_s's through $\overline{p}$. We prove more than this in Lemma (3.3). The lemma in fact implies that at least $(g-5)$-dimensional family of Λ_s's pass through a point $\overline{p} \in C_\eta$.

Lemma 3.3. *For any line bundle $L \in \mathrm{Sing}(\Xi)$, there exists a surface $Y \subset \tilde{\alpha}^{-1}(L)$ such that through any point $\overline{p}$ of C_η it passes 2-dimensional family of Λ_s's parameterized by Y, through $\overline{p}$.*

Using Lemma (3.3), for a line bundle $L \in \mathrm{Sing}(\Xi)$, one can interpret the linear spaces contained in Q_L parametrrized by $\tilde{\alpha}^{-1}(L)$, as linear spaces passing through the points of C_η. Therefore although the linear spaces inside the projectivized tangent cone of the prym variety at the points $L \in \mathrm{Sing}(\Xi)$, which are parametrized by $\tilde{\alpha}^{-1}(L)$, are of high codimension; but any member of this family cuts the curve C_η at least in one point. Summarizing we have:

Lemma 3.4. *Any member of family (1), Λ_s, cuts the curve C_η at least in one point.*

Consider that Lemma (3.3) implies that the focal map of the family (1), drops rank twice at any point $p \in C_\eta \cap F_s$. Now by this together with results of the paper [7], it is seen immediately that:

Theorem 3.5. *For any $s \in S$ the focal scheme of the family (1) at $s \in S$, F_s, is a singular scheme. Precisely the set $C_\eta \cap F_s$, which by Lemma (3.3) is a nonempty set, is contained in* $\mathrm{Sing}(F_s)$.

4 Reference

References

[1] A. Bajravani, Focal Varieties of Curves of genus 6 and 8, Rendiconte del Circolo Matematico di Palermo, 2010.

[2] A. Bajravani; E. Sernesi; A Note on Projectivized Tangent Cone Quadrics of Rank 4 in the Ideal of a Prym-Canonical Curve, To appear.

[3] C. Ciliberto, E. Sernesi, Projective Geometry related to the Singularities of Theta divisors of Jacobians, Bollettino dell'Unione Matematica Italiana, (9) III, 93-109, (2010).

[4] C. Ciliberto, E. Sernesi, Singularities of Theta divisors and congruences of Planes, Journal of Algebraic Geometry, 1, 231-250 (1992).

[5] C. Ciliberto, E. Sernesi, Singularities of Theta Divisor and Families of Secant Spaces to a Canonical Curve, J. Algebra, 171, 867-893 (1995).

[6] C. Ciliberto, E. Sernesi, On the Geometry of Canonical Curves of Odd Genus, Communications in Algebra, 28(12) (2000), 5993–6001.

[7] D. Eisenbud, Linear Sections of Determinantal Varieties, Amer. J. Math., 110 (1988), 541-575.

[8] R. Smith, R. Varley; A Riemann Singularities Theorem for Prym Theta Divisors, with applications, Pacific Journal of Mathematics, volume 201, No. 2, 479-509, December 2001.

[9] R. Smith; R. Varley; The Curve of "Prym Canonical" Gauss Divisors on a Prym Theta Divisor, Transactions of the A. M. S. V.353, N.12, pages 4949-4962(2001).

Books:

[10] E. Arbarello, M. Cornalba, Ph. Griffiths, J. Harris, Geometry of Algebraic Curves. Vol. I, volume 267 of Grunlehren der Mathematischen Wissenschaften. Springer-Verlag, New York, 1985.

[11] Ch. Birchenhake, H. Lang; Complex Abelian Varieties, volume 302 of Grunlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, Second Edition, 2004.

[12] R. Hartshorne; Algebraic Geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No.52.

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Randers metrics of Douglas type on 4-dimensional hypercomplex Lie groups

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Abstract: In this paper we construct Randers metrics of Douglas type on 4-dimensional hypercomplex Lie groups. Then we give their flag curvature formulae explicitly as a generalization of Randers metrics of Berwald type which imply examples of locally projectively flat Finsler spaces. The set of all homogeneous geodesics on these spaces with left invariant Randers metrics of Douglas type are also investigated.

Keywords: Homogeneous Randers spaces, Flag curvature, Douglas type, Homogeneous geodesics.

1 INTRODUCTION

Finsler metrics have many important applications in Physics and biology(see [3]). A very interesting type of Finsler metrics are Randers metrics which for the first time were introduced by physicist Randers in 1941.

A Randers metric F on a connected smooth manifold M can be written as $F = \alpha + \beta$, where α is the underling Riemannian metric on M and β is an smooth 1-form on M which for all $x \in M$ the length of the form β with respect to α is less than one or $\parallel \beta \parallel_x \prec 1$(see [7]). Since the Riemannian metric α induces a bijection between 1-forms and vector fields on M, the 1-form β corresponds to a vector field V on M. If the vector filed V is parallel with respect to the Riemannian metric α, then the Randers metric F is said to be of Berwald type and if the 1-form β is closed, then the Randers metric F is said to be of Douglas type. Douglas metrics are more generalized than Berwald metrics(see [5]). In [1] we have generalized some results from the Randers metrics of Berwald type to the Randers metrics of Douglas type. Our aim in this paper is to extend Berwald metrics on 4-dimensional hypercomplex Lie groups.

Hypercomplex structures appear in the study of black holes and sigma model spaces and they have many application in physics. In [4] Barberis classified 4-dimensional hypercomplex Lie groups. Then Salimi Moghaddam in [12] gave explicit formulae for computing sectional curvatures on these spaces. Later on he generalized his study for Randers metrics of Berwald type on these spaces and in [13] he gave explicit formulae for computing flag curvature of these metrics. In this paper we extend his study for the Randers metrics of Douglas type. In fact we first equipped these Lie groups with a left invariant Randers metrics of Douglas type, then we obtain their flag curvature formulae explicitly. We also investigate the set of all homogeneous geodesics on these spaces with a left invariant Randers metric of Douglas type.

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2 On 4-Dimensional Hypercomplex Lie groups

Here we remind the classification of 4-dimensional hypercomplex Lie groups given in [4].

Theorem 2.1. *Let G be a hypercomplex 4-dimensional simply connected Lie group with a left invariant Hermitian metric g. Then $\mathcal{G}$ is isomorphic to one of the following Lie algebras:*
(1) $[Y,X]=[Z,X]=[W,X]=[Y,Z]=[Y,W]=[Y,W]=0$,
(2) $[Y,Z]=W,\ [Z,W]=Y,\ [W,Y]=Z$, *and* X *is central,*
(3) $[X,Z]=X,\ [Y,Z]=Y,\ [X,W]=Y,\ [Y,W]=-X$,
(4) $[X,Y]=Y,\ [X,Z]=Z,\ [X,W]=W$,
(5) $[X,Y]=Y,\quad [X,Z]=\frac{1}{2}Z,\quad [X,W]=\frac{1}{2}W,\quad [Z,W]=\frac{1}{2}Y$,
where $\{X,Y,Z,W\}$ *is an orthonormal basis.*

Recall that if G is a Lie group with a left invariant Randers metric F, where F is defined by the Riemannian metric $a=a_{ij}dx^i\otimes dx^j$ and the vector field V, then F is of Douglas type if and only if V satisfies

$$<[m,n],V>=0,\ \textit{for all}\ m,n\in\mathcal{G}, \qquad (1)$$

for more details see p.94 theorem 3.2 in [2]. Also more exactly a Randers metric F on a smooth n-dimensional manifold M can be written as

$$F(x,y)=\sqrt{\alpha_x<y,y>}+\alpha_x<y,V>, \qquad (2)$$

where $\alpha=a_{ij}dx^i\otimes dx^j$ is the Riemannian metric, $x\in M$, $y\in T_xM$ and $\alpha_x<V,V>\prec 1$(see [8]).

Theorem 2.2. *The Randers metrics of Douglas type on 4-dimensional hypercomplex Lie groups are given as follows*
(1) $F(s)=\sqrt{\sum_{i=1}^4 a_i^2}+K_1a_1+K_2a_2+K_3a_3+K_4a_4$, *where* $K_1^2+K_2^2+K_3^2+K_4^2\prec 1$,
(2) $F(s)=\sqrt{\sum_{i=1}^4 a_i^2}+K_1a_1$, *where* $|K_1|\prec 1$,
(3) $F(s)=\sqrt{\sum_{i=1}^4 a_i^2}+K_3a_3+K_4a_4$, *where* $K_3^2+K_4^2\prec 1$,
(4) $F(s)=\sqrt{\sum_{i=1}^4 a_i^2}+K_1a_1$, *where* $|K_1|\prec 1$,
(5) $F(s)=\sqrt{\sum_{i=1}^4 a_i^2}+K_1a_1$, *where* $|K_1|\prec 1$,
where $s=a_1X+a_2Y+a_3Z+a_4W$ *is a vector in the lie algebra* $\mathcal{G}$ *of* G.

Proof. In the case 1 of theorem 2.1 by using Eq.(1) where $V=K_1X+K_2Y+K_3Z+K_4W$ is a vector in $\mathcal{G}$ and $m,n\in\mathcal{G}$ are orthonormal, we have $V=K_1X+K_2Y+K_3Z+K_4W$. If we replace V in Eq.(2) we have $F(s)=\sqrt{\sum_{i=1}^4 a_i^2}+K_1a_1$. Also since F is a Randers metric we have $K_1^2+K_2^2+K_3^2+K_4^2\prec 1$. In other cases in the same way we obtain the desired result. □

Theorem 2.3. *Let G be a four dimensional hypercomplex Lie group, which is equipped with a Randers metric F of Douglas type. Also let (P,s) be a flag in $\mathcal{G}$ such that $\{t,s\}$ is an orthonormal basis of P with respect to the left invariant Riemannian metric g. Then the flag curvature formulae on 4-dimensional hypercomplex Lie groups in 5-cases respectively are given as follow.*
Case 1:

$$K(P,s)=0, \qquad (3)$$

Case 2:

$$K(P,s)=\frac{(a_2b_3-a+3b_2)^2+(a_2b_4-a_4b_2)^2}{4(1+K_1a_1}+\frac{(a_3b_4-a_4b_3)^2}{4(1+K_1a_1)^2}, \qquad (4)$$

Case 3:

$$K(P,s)=\frac{-\{(a_1b_2-b_2a_1)(aY-bX)}{(1+K_3a_3+K_4a_4)^2}+\frac{(a_1b_3-a_3b_1)(a_1Z-a_3X)}{(1+K_3Z+K_4W)^2}+\frac{(a_2b_3-a_3b_2)(a_2Z-a_3Y)\}}{(1+K_3a_3+K_4a_4)^2}+\frac{3((a_1^2+a_2^2)K_3)^2}{4(1+K_3a_3+K_4a_4)^4}+\frac{K_3a_3(a_1^2+a_2^2)}{(1+K_3a_3+K_4a_4)^3}, \qquad (5)$$

Case 4:

$$K(P,s)=\frac{-\{(a_1b_2-a_2b_1)^2+(a_1b_3-a_3b_1)^2}{(1+K_1X)^2}$$

$$+\frac{(a_2b_3-a_3b_2)^2+(a_1b_4-a_4b_1)^2}{(1+K_1X)^2}$$
$$+\frac{(a_2b_4-b_2a_4)^2+(a_3b_4-a_4b_3)^2\}}{(1+K_1X)^2}$$
$$+\frac{3(K_1(a_2^2+a_3^2+a_4^2))^2}{4(1+K_1X)^4}+\frac{K_1a_1(a_2^2+a_3^2+a_4^2)}{(1+K_1X)^3}, \qquad (6)$$

Case 5:

$$K(P,s)=\frac{-\{a_1b_2-a_2b_1+\frac{1}{4}(a_3b_4-a_4b_3))^2}{(1+K_1a_1)^2}$$
$$+\frac{\frac{1}{4}(a_1b_3-a_3b_1+\frac{1}{2}(a_2b_4-b_4a_2))^2}{(1+K_1a_1)^2}$$
$$+\frac{\frac{1}{4}(a_1b_4-b_4a_1-\frac{1}{2}(a_2b_3-a_3b_2))+\frac{3}{8}((a_2b_3-b_2a_3)^2}{(1+K_1a_1)^2}$$
$$+\frac{(a_2b_4-a_4b_2)^2+(a_3b_4-a_4b_3)^2\}}{(1+K_1a_1)^2}$$
$$+\frac{3((\frac{2a_2^2+a_3^2+a_4^2}{2})K_1)^2}{4(1+K_1a_1)^4}+\frac{K_1a_1(\frac{4a_2^2+a_3^2+a_4^2}{4})}{(1+K_1a_1)^3}. \qquad (7)$$

Proof. In the case 1 of theorem 2.1 by using the following equation

$$2<U(X,Y),Z>=<[Z,X],Y>+<[Z,Y],X>, \qquad (8)$$

for all $X,Y,Z\in\mathcal{G}$, we have

$$U(s,t)=0, \qquad (9)$$

where $s=a_1X+a_2Y+a_3Z+a_4W$ and $t=b_1X+b_2Y+b_3Z+b_4W$. We can use the following formula which is given in theorem 2.1, p.72 in [6] for calculating the flag curvature of the Randers metrics of Douglas type;

$$K(P,s)=\frac{\alpha^2}{F^2}\bar{K}(P)+\frac{1}{4F^4}(3<U(y,y),V>^2$$
$$-4F<U(y,U(y,y)),u>), \qquad (10)$$

where $\bar{K}(P)$ is the sectional curvature of the Riemannian metric g. If we apply Eq.(9), then we have Eq.(3).
In the case 2 of theorem 2.1 by using Eq.(11) we have

$$U(s,t)=U(s,s)=U(t,t)=0,$$

where $s=a_1X+a_2Y+a_3Z+a_4W$ and $t=b_1X+b_2Y+b_3Z+b_4W$. Also the sectional curvature is given in [12]. If we replace them in Eq.(10), then we have Eq.(4).
In the case 3 of theorem 2.1 similar to case 1 we get

$$U(s,t)=\frac{1}{2}(b_1a_3+a_4b_2+a_1b_3+a_2b_4)X+$$
$$\frac{1}{2}(-a_4b_1+a_3b_2-a_1b_4+a_2b_3)Y+(-a_1b_1-a_2b_2)Z,$$

where $s=a_1X+a_2Y+a_3Z+a_4W$ and $t=b_1X+b_2Y+b_3Z+b_4W$. Then we have $<U(s,s),V>=-K_3(a_1^2+a_2^2)$ and $<U(s,U(s,s)),V>=-(a_1^2+a_2^2)a_3K_3$. Also the sectional curvature $\bar{K}$ is given in [12]. If we replace them in Eq.(10) we have Eq.(5).
In the case 4 of theorem 2.1 since

$$U(s,t)=(a_2b_2+a_3b_3+a_4b_4)X+\frac{-1}{2}(a_1b_2+b_1a_2)Y$$
$$-\frac{1}{2}(a_1b_3+b_1a_3)Z-\frac{1}{2}(a_1b_4+b_1a_4)W,$$

where $s=a_1X+a_2Y+a_3Z+a_4W$ and $t=b_1X+b_2Y+b_3Z+b_4W$. Then we have $<U(s,s),V>=(a_2^2+a_3^2+a_4^2)K_1$ and $<U(s,U(s,s)),V>=-K_1a_1(a_2^2+a_3^2+a_4^2)$. Also the sectional curvature $\bar{K}$ is given in [12]. If we replace them in Eq.(10) we have Eq.(6).
In the case 5 of theorem 2.1 by using the following equation

$$2<U(X,Y),Z>=<[Z,X],Y>+<[Z,Y],X>, \qquad (11)$$

for all $X,Y,Z\in\mathcal{G}$, we have

$$U(s,t)=(\frac{2b_2a_2+b_3a_3+b_4a_4}{2})X+$$
$$(\frac{-b_1a_2-a_1b_2}{2})Y+(\frac{b_4a_2+a_4b_2-b_1a_3-a_1b_3}{4})Z$$
$$+(\frac{-b_3a_2-a_3b_2-b_1a_4-a_1b_4}{4})W,$$

where $s=a_1X+a_2Y+a_3Z+a_4W$ and $t=b_1X+b_2Y+b_3Z+b_4W$. Then we have $<U(s,s),V>=\frac{2a_2^2+a_3^2+a_4^2}{2}K_1$ and $<U(s,U(s,s)),V>=\frac{-a_1(4a_2^2+a_3^2+a_4^2}{4}$. Also the sectional curvature $\bar{K}$ is given in [12]. By replacing them in Eq.(10) we have the desired result. □

Here we remind the following theorem from [11].

Theorem 2.4. *A finsler metric F on a manifold M ($dimM \geq 3$) is locally projectively flat if and only if F is a Douglas metric with scalar flag curvature.*

So by theorems 2.6 and 2.4 we have the following result.

Corollary 2.5. *Every hypercomplex four dimensional Lie group with the Randers metrics of Douglas type is a locally projectively flat Finsler spaces.*

Here we state the exact form of homogeneous geodesics on hypercomplex four dimensional Lie groups.

Theorem 2.6. *Let G be a four dimensional hypercomplex Lie group, which is equipped with a Randers metric F of Douglas type. Then y is a geodesic vector of (G_n, F) if and only if y has one of the following forms*

(1) $y = a_1X + a_2Y + a_3Z + a_4W$
(2) $y = a_1X + a_2Y + a_3Z + a_4W$
(3) $y = a_3Z + a_4W$
(4) $y = a_1X$
(5) $y = a_1X$

Proof. By the following formula in [9, 10], for the case 4 we have

$$< a_1X_1 + \frac{a_1X + a_2Y + a_3Z + a_4W}{\sqrt{\sum_{i=1}^{4} a_i^2}}, [a_1X + a_2Y + a_3Z + a_4W, e_j] >= 0,$$

where $e_j = X, Y, Z$ or W which implies that $y = a_1X$. The proof of (1), (2), (3) and (5) are similar. □

References

[1] M. Aghasi, and M. Nasehi, "On homogeneous Randers spaces with Douglas or naturally reductive metrics," submitted.

[2] H. An, and S. Deng, "Invariant (α, β)-metric on homogeneous manifolds," Monatsh math. vol. 154, pp. 89-120, Jun. 2008.

[3] P. L. Antonelli, R. S. Ingarden, and M. Matsumoto, "The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology", FTPH Vol. 58, Kluwer Academic Publishers, 1993.

[4] M. L. Barberis, "Hypercomplex structures on 4-dimensional Lie groups," Proc. Amer. Math. Soc, vol. 125, pp. 1043-1054, Apr. 1997.

[5] X. Chen, and Z. Shen, "On Douglas metrics," Publ Math Debrecen, vol. 66, pp. 503-512, Feb. 2005.

[6] S. Deng, and Z. Hu, "On flag curvature of homogeneous Randers spaces," Canad. J. Math, vol. 65, pp. 66-81, Feb. 2013.

[7] S. Deng, "The S-curvature of homogeneous Randers spaces", Differ. Geom. Appl, vol. 27, pp. 7584, Feb. 2009.

[8] S. Deng, Z. Hou, "Invariant Randers Metrics on Homogeneous Riemannian Manifolds," J. Phys. A: Math. Gen, vol. 37, pp. 4353-4360, Mar. 2004.

[9] D. Latifi, "Homogeneous geodesics in homogeneous Finsler spaces," J. Geom. Phys. vol. 57, pp. 1421-1433, Apr. 2007.

[10] D. Latifi, A. Razavi, "Homogeneous geodesics of left invariant Randers metrics on a three dimensional Lie groups,", Int. J. Contemp. Math. Sciences, vol. 4, pp. 873-881, 2009.

[11] H. Liu, S. Deng, "Homogeneous (α, β)-metrics of Douglas type," Forum Math. pp. 1-17, Feb. 2014.

[12] H. R. Salimi Moghaddam, "On some hypercomplex 4-Dimensional Lie groups of constant scalar curvature," Int. J. Geom. Methods Mod. Phys, vol. 6, pp. 619-624, Jun. 2009.

[13] H. R. Salimi Moghaddam, "Randers metrics of Berwald type on 4-Dimensional hypercomplex Lie groups," J. Phys. A: Math. Theor, vol. 42, pp. 1-7, Feb. 2009.

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G-MANIFOLDS WITH NEGATIVE CURVATURE

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Abstract: We classify cohomogeneity two UND-Riemannian G-manifolds under the condition $M^G \neq \emptyset$
Keywords: Lie group ; Isometry ; Manifold.

1 INTRODUCTION

A classic theorem about Riemannian manifolds of non-positive curvature ([16]) states that a homogeneous Riemannian manifold M of non positive curvature is simply connected or it is diffeomorphic to a cylinder over a torus (i.e, it is diffeomorphic to $R^k \times T^s$, $k + s = dimM$). It is interesting to reduce the homogeneity condition to weaker conditions and see what happens to the topology of M. When M is homogeneous then there is a connected and closed subgroup G of the isometries of M such that the orbit space of the action of G on M, $\frac{M}{G}$ is a one point set. A weaker condition is that $dim\frac{M}{G} = 1$ or 2 (i.e, M be a cohomogeneity one or cohomogeneity two G-manifold). There are some interesting theorems about topological properties of cohomogeneity one G-manifolds of non-positive curvature under conditions on G and M (see [1],[12], [13], [15]). There is a topological characterization of cohomogeneity one UND-Riemannian manifolds (Riemannian manifolds with the property that the universal covering manifold decomposes as a direct product of negatively curved manifolds) in [13]. Following the papers [9-11], where the first author proved various results about topological properties of cohomogeneity two negatively curved G-manifold M under some special conditions on M or G, we are going to consider some cohomogeneity two UND- manifolds in the present paper. We topologically characterize a UND-manifold M which is acted on isometrically by a connected and closed subgroup G of isometries, under the condition that the fixed point set of the action is not empty

References

[1] Abedi H., Alekssevsky D. V., Kashani S. M. B., *Cohomogeneity one Riemannian manifolds of non-positive curvature*, Differential Geometry and its Applications, 25(2007) 561-581.

[2] Bishop R. L.O'Neill B., *Manifolds of negative curvature*, Trans. Amer. Math. Soc. 145 (1969) 1-49.

[3] Bredon. G. E., *Introduction to compact transformation groups*, Acad. Press, New york, London 1972.

*Corresponding Author
†Corresponding Author

[4] Do Carmo M. P., *Riemannian geometry*, Brikhauser, Boston, Basel, Berlin, 1992.

[5] Eberlin P. and O'Neil B., *Visibility manifolds* , Pasific J. Math. 46 (1973) 45-109.

[6] Eberlein P., *Geodesic flows in manifolds of nonpositive curvature*, $http://math.unc.edu/Faculty/pbe/AMS\_Summer.pdf$.

[7] Kobayashi S., *Homogeneous Riemannian manifolds of negative curvature*, Toho. Math. J. 14, 413-415(1962).

[8] Michor P.W., *Isomrtric actions of Lie groups and invariants* , Lecture course at the university of Vienna, 1996/97, $http://www.mat.univie.ac.at/{\sim}michor/tgbook.ps$.

[9] Mirzaie R., *On negatively curved G-manifolds of low cohomogeneity*, Hokkaido Mathernatical Journal Vol. 38, 797-803(2009).

[10] Mirzaie R., *On Riemannian manifolds of constant negative curvature*, J. Korean Math. Soc. 48, No. 1, 23-31(2011).

[11] Mirzaie R., *On homoheneous submanifolds of negatively curved Riemannian manifolds*, Publ. Math. Debrecen 82 (2013) 267-275.

[12] Mirzaie R. and Kashani S. M. B., *On cohomogeneity one flat Riemannian manifolds*, Glasgow Math. J. 44 (2002) 185190.

[13] Mirzaie, R. and Kashani S. M. B., *Topological properties of some cohomogeneity one Riemannian manifolds of non-positive curvature*, Bull. Korean Math. Soc. 37 (2000), No. 3, pp. 587-599.

[14] O'Neill B., *Semi Riemannian geomerty with applications to Relativity*, Academic press, New york, Berkeley 1983.

[15] Podesta F. and Spiro A., *Some topological propetrties of cohomogeneity one Riemannian manifolds with negative curvature*, Ann. Global Anal. Geom. 14 69-79(1996).

[16] Wolf J. A., *Homogeneity and bounded isometries in manifolds of negative curvature,* Illinos J. Math. **8** (1964) 14-18.

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Lie symmetry classification of $u_t - u_{x^2t} + u_x f(u) - a u_x u_{x^2} - b u u_{x^3} = 0$

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Abstract: In this paper, we investigate a symmetry classification of a nonlinear third order partial differential equation $u_t - u_{x^2t} + u_x f(u) - a u_x u_{x^2} - b u u_{x^3} = 0$ where $f(u)$ is smooth function on u and a, b are arbitrary constans. Three special cases of this equation, using Lie group method , is given.

Keywords: Lie symmetry analysis, Adjoint representation, nonlinear third order partial differential equation.

1 INTRODUCTION

The theory of Lie groups of differential equations was developed by Sophus Lie. In this paper, we study the following third-order nonlinear equation

$$u_t - u_{x^2t} + u_x f(u) - a u_x u_{x^2} - b u u_{x^3} = 0 \qquad (1)$$

There are three cases to consider: 1) $b \neq 0$, $a =$ arbitrary constant, 2) $b = 0$, $a \neq 0$, and 3) $b = 0$, $a = 0$. In [?], Clarkson, Mansfield and Priestly are concerned with symmetry reductions of the nonlinear third order partial differential equation given by

$$u_t - \epsilon u_{x^2t} + (k - u)u_x - u u_{x^3} - \beta u_x u_{x^2} = 0, \quad (2)$$

Where ϵ, k, and β are arbitrary constants. The special cases of (??) are:

i) Cammasa-Holm (CH) equation $u_t - u_{x^2t} + (k + 3u)u_x = u u_{x^3} + 2u_x u_{x^2}$, k -arbitrary (real), describing the unidirectional propagation of shallow water waves over a flat bottom (let$f(u) = k + 3u$, $a = 2$, $b = 1$ in (??).

ii) Degas peris-Procesi (DP) equation $u_t - u_{x^2t} + (k + 4u)u_x = u u_{x^3} + 3u_x u_{x^2}$, k -arbitrary (real), is another equation of this class (let $f(u) = k + 4u$, $a = 3$, $b = 1$ in (??).

iii) Fornberg whitham (FW) equation $u_t - u_{x^2t} + (1 + u)u_x = u u_{x^3} + 3u_x u_{x^2}$, is another equation of this class (let$f(u) = 1 + u$, $a = 3$, $b = 1$ in (??)).

iv) BBM equation $u_t - u_{x^2t} + u_x + (uu)_x = 0$, is another equation of this class (let $f(u) = 1 + u$, $a = 0$, $b = 0$ in (??)).

2 Method of Lie symmetry

In this section, we recall the general procedure for determining symmetries for an arbitrary system of partial differential equations [?]. Let us consider the general system of a nonlinear system of partial differential equations of order n, containing p independent and q dependent variables is given as follows

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots \qquad (3)$$

Involving $x = (x^1, \cdots, x^p)$, $u = (u^1, \cdots, u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. The one-parameter Lie group of transformations of the system (??): $\bar{x}_i = x^i + \epsilon\xi^i(x,u) + O(\epsilon^2)$, $\bar{u}_j = u^j + \epsilon\phi^j(x,u) + O(\epsilon^2)$, Where$i = 1,\cdots,p$,$j = 1,\cdots,q$ and ξ^i,ϕ^j are the infinitesimal of the transformations for the independent and dependent variables, respectively and ϵ is the transformation parameter. We consider the general vector field v as the infinitesimal generator associated with the above group $v = \sum_{i=1}^{p} \xi^i(x,u)\,\partial_{x^i} + \sum_{j=1}^{q} \phi^j(x,u)\,\partial_{u^j}$. A symmetry of a differential equation is a transformation, which maps solutions of the equation to other solutions. The invariance of the system (??) under the infinitesimal transformation leads to the invariance conditions. (Theorem 2.36 of [?], Theorem 6.5 of [?]).

$$v^n[\Delta_\nu(x,u^n)] = 0, \quad \Delta_\nu(x,u^n) = 0, \quad \nu = 1,\cdots,r, \tag{4}$$

where v^n is called the n^{th} order prolongation of the infinitesimal generator given by $v^n = v + \sum_{j=1}^{q}\sum_k \phi_k^j(x,u^{(n)})\,\partial_{u_k^j}$, where $k = (i_1,\cdots i_\alpha)$, $1 \le i_\alpha \le p$, $1 \le \alpha \le n$, and the sum is over all k's of order $0 < \#k \le n$. If $\#k = \alpha$, the coeficent ϕ_k^j of $\partial_{u_k^j}$, will depend onlly on α'th and lower order derivatives of u and? $\phi_j^k(x,u^n) = D_k(\phi_j - \sum_{i=1}^{p}(\xi^i u_i^j)) + \sum_{i=1}^{p}\xi^i u_{k,i}^j$, where $u_i^j := \partial u^j/\partial x^i$ and $u_{k,i}^j := \partial u_k^j/\partial x^i$.There are tree cases to consider.

3 Classical Symmetry Method

In this section, we will perfom Lie group method for Eq.(??). on $(x^1 = x, x^2 = t, u^1 = u)$, $(\tilde{x},\tilde{t},\tilde{u}) = (x,t,u) + \epsilon(\xi(x,t,u),\tau(x,t,u),\phi(x,t,u)) + O(\epsilon^2)$, where $\epsilon \le 1$ the group parameter and $\xi^1 = \xi$, $\xi^2 = \tau$ and$\phi^1 = \phi$ are the infinitesimals of the transformations for the independent and dependent variables respectively, the associated vector fields is of the form $v = \xi(x,t,u)\,\partial_x + \tau(x,t,u)\,\partial_t + \phi(x,t,u)\,\partial_u$ and the third porolongation of v is the vector field

$$v^{(3)} = v + \phi^x\,\partial_{u_x} + \phi^t\,\partial_{u_t} + \phi^{x^2}\,\partial_{u_{x^2}} + \phi^{xt}\,\partial u_{xt} + \cdots + \phi^{ttt}\,\partial u_{ttt}, \tag{5}$$

with coefficent

$$\phi^k = D_k(\phi - \sum_{i=1}^{2}\xi^i u_i^j) + \sum_{i=1}^{2}\xi^i u_{k,i} \tag{6}$$

where D_k is the total derivative with respect to independent variables. By Theoreom2.36 in [?], the invariance condition for Eq. (??) is given by,

$$v^{(3)}[u_t - u_{x^2t} + u_x f(u) - a u_x u_{x^2} - b u u_{x^3}] = 0, \tag{7}$$

whenever $[u_t - u_{x^2t} + u_x f(u) - a u_x u_{x^2} - b u u_{x^3}] = 0$. the condition ?? is equivalent to

$$\phi^t - b u_{x^3}\phi + (f(u) - a u_{x^2})\phi^x - a u_x \phi^{x^2} - b u \phi^{x^3} - \phi^{x^2 t} = 0, \tag{8}$$

whenever $[u_t - u_{x^2t} + u_x f(u) - a u_x u_{x^2} - b u u_{x^3}] = 0$. substituting (??) into invariance condition (??), yields the determining equations. there are three cases to consider.

3.1 $b \neq 0$, $a =$arbitrary constant.

Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

$$v_1 = -t\partial_x + t\partial_t - \frac{(bu+1)}{b}\partial_u, \quad v_2 = \partial_t, \quad v_3 = \partial_x. \tag{9}$$

we have

$$f(u) = -1 + K(bu+1) \tag{10}$$

to compute the Adjoint representation, we use the Lie series

$$Ad(exp(\epsilon v_i)v_j) = v_j - \epsilon[v_i,v_j] + \frac{\epsilon^2}{2}[v_i,[v_i,v_j]] - \cdots, \tag{11}$$

where $[v_i,v_j]$ is the commutator for the Lie algebra,ϵ is a parameter, and $i,j = 1,2,3$, then we have the table 1.

TABLE 1

Adjoint representation table of the infinitesimal generators in this case

case	v_1	v_2	v_3
v_1	v_1	$e^{-\epsilon}v_2$	$(-e^{\epsilon}+1)v_2 + v_3$
v_2	v_1	$\epsilon v_1 + v_2$	$-\epsilon v_1 + v_3$
v_3	v_1	v_2	v_3

3.2 $a \neq 0,\ b = 0$

to find complete solution of the determining system, we consider $l = dimSpam_R\{f_u, f, l\}$ three cases are possible

i) $l = 1$, then $f = constant$,

ii) $l = 2$, then $f_u = \alpha f + \beta$,

iii) $l = 3$, then $\alpha f_u + \beta f + \gamma \neq 0, \quad \alpha \neq 0$.

Case i) In this case with substituting $f = constant$ in determining system the Infinitesimal generators of every one parameter Lie group of point symmetries are:

$$v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = \partial_u. \tag{12}$$

Case ii) In this case with integrating from $f_u = \alpha f + \beta$ with respect to u we obtain

$$f(u) = \frac{-\beta}{\alpha} + Ce^{\alpha u} \tag{13}$$

where C is an integrating constant. Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

$$v_1 = -t\partial_t + \frac{c_2(C\alpha e^{\alpha u} - \beta)}{C\alpha^2 e^{\alpha u}}\partial_u, \quad v_2 = \partial_t, \quad v_3 = \partial_x. \tag{14}$$

the Adjoint representation table of the infinitesimal generators v_i is given in table 2.

TABLE 2
Adjoint representation table of the infinitesimal generators in this case

case	v_1	v_2	v_3
v_1	v_1	$e^{\epsilon}v_2$	v_3
v_2	v_1	$-\epsilon v_1 + v_2$	v_3
v_3	v_1	v_2	v_3

Case iii) Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

$$v_1 = \partial_t, \quad v_2 = \partial_x. \tag{15}$$

3.3 $b = 0,\ a = 0.$

in this case, to find a complete solution of the complete set of determining equation with assumption $f_u \neq 0$ two general cases are possible:

$$i)\ \frac{f}{f_u} = c, \qquad ii)\ \frac{f}{f_u} = h(u).$$

where c is constant.

Case i) In this case with integrating from $\frac{f}{f_u} = c$ with respect to u we have

$$f(u) = Le^{\frac{u}{c}}. \tag{16}$$

where L is an integrating constant. Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

$$v_1 = -t\partial_t - c\partial_u, \quad v_2 = \partial_t, \quad v_3 = \partial_x. \tag{17}$$

the Adjoint representation table of the infinitesimal generatorsv_i is given in table 3

TABLE 3
Adjoint representation table of the infinitesimal generators in this case

case	v_1	v_2	v_3
v_1	v_1	$e^{\epsilon}v_2$	v_3
v_2	v_1	$\epsilon v_1 + v_2$	v_3
v_3	v_1	v_2	v_3

Case ii) In this case, we find that the components ξ, τ and ϕ are $\xi = \xi(x)$, $\tau = \tau(t)$ and $\phi = A(x)u + B(x,t)$. Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

$$v_1 = t\partial_t, \quad v_2 = \partial_t. \tag{18}$$

References

[1] Clarkson P A, Mansfield E L and Priestley T J, Symmetries of a Class of Nonlinear Third Order Partial Differential Equations, Math. Comput. Modelling 25 (1997), 195-212.

[2] P.J. Olver, Applications of Lie Group for Differential Equations?, ?Springer-Verlag, New York, 1986.

[3] P.J. Olver, Equivalence, invariant and symmetry, Cambridge University Press, Cambridge 1995.

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frames of rank 0 on local top spaces

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Abstract: It is proved that Maurer-Cartan frame exist on local top spaces with finite number of identities. It is also proved that for every local top space with finite number of identities, there is a covering top space which is also a local covering of an open subset M of a global Lie group G.
Keywords: Lie group, Lie algebra, Local Lie group, top space, local top space.

1 INTRODUCTION

The definition of local Lie group is generalized in [1], by using of top spaces[2]. Let us recall the definition of a local Lie group and its generalization, local top spaces.

Definition 1.1[3] A smooth manifold L is called a local Lie group if there exists

- a distinguished element $e \in L$, the identity element,
- a smooth product map $\mu : U \to L$ defined on an open subset $(\{e\} \times L) \cup (L \times \{e\}) \subset U \subset (L \times L)$,
- a smooth inversion map $i : V \to L$ defined on an open subset $e \in V \subset L$ such that $V \times i(V) \subset U$, and $i(V) \times V \subset U$,

all satisfying the following properties:

(i) Identity: $\mu(e, x) = x = \mu(x, e)$ for all $x \in L$,

(ii) Inverse: $\mu(i(x), x) = \mu(x, i(x)) = e$ for all $x \in V$,

(iii) Associativity: If $(x, y), (y, z), (\mu(x, y), z)$ and $(x, \mu(y, z))$ all belongs to U, then

$$\mu(\mu(x, y), z) = \mu(x, \mu(y, z)).$$

Definition 1.2. A smooth manifold H is called a local top space if there exists

- a set $e(H) \subset H$, the identity elements,
- a smooth product map $\mu : U \to H$ defined on an open subset $(e(H) \times H) \cup (H \times e(H)) \subset U \subset (H \times H)$,
- a smooth inversion map $i : V \to H$ defined on an open subset $e(H) \subset V \subset H$ such that $V \times i(V) \subset U$, and $i(V) \times V \subset U$,

all satisfying the following properties:

(i) Identity: For each $x \in H$ there is a unique element $e(x)$ such that $\mu(e(x), x) = x = \mu(x, e(x))$.

(ii) Inverse: $\mu(i(x), x) = \mu(x, i(x)) = e(x)$ for all $x \in V$.

(iii) Associativity: If $(x, y), (y, z), (\mu(x, y), z)$ and $(x, \mu(y, z))$ all belongs to U, then

$$\mu(\mu(x, y), z) = \mu(x, \mu(y, z)).$$

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(iv) $\mu(e(x), e(y)) = e(\mu(x, y))$, for each $x, y \in H$.

(ivv) $e : H \to H$ is a smooth map.

One can use the symbol (H, μ, U, i, V) for a local top space H with the functions μ, i, U and V as the above definition.

Example 1.1. Let $H \subseteq T$ be a neighborhood of $e(T)$ in the top space T and U be any open subset of $H \times H$ such that $(e(T) \times H) \cup (H \times e(T)) \subset U \subset (H \times H) \cap \mu^{-1}(H)$. In addition let V be any open subset of H such that $e(T) \subset V \subset H \cap i^{-1}(H)$, and $(V \times i(V)) \cup (i(V) \times V) \subset U$. The group multiplication μ and inversion i on T then restricted to define local top space multiplication and inversion maps on H.

Definition 1.3. Let M be a smooth m-dimensional manifold. A frame on M is a set of vector fields $V = \{v_1, ..., v_m\}$ such that $\{v_1(x), ..., v_m(x)\}$ form a basis for the tangent space TM_x at each point $x \in M$.

Remark 1.1. Let V be a frame for M. There exists functions C^k_{ij} such that:

$$[v_i, v_j] = \Sigma C^k_{ij} v_k.$$

The functions C^k_{ij} are called structure functions of the given frame. The frame V is said to have rank 0 if the functions C^k_{ij} are all constant. For example the structure functions of the Lie algebra g are constant.

Left and right multiplication with x are defined with:

$$l_x(y) = \mu(x, y), \qquad r_x(y) = \mu(y, x)$$

Definition 1.4.[3] A local Lie group L is regular if, for each $x \in L$, the maps l_x and r_x are diffeomorphisms on their respective domains of definition.

Definition 1.5. A local top space H is regular if $e^{-1}(e(x))$ is regular local Lie group, for every $x \in H$.

Definition 1.6.[3] A local top space H is associative to order n if, for every $3 \leq m \leq n$, and every ordered $m-$tuple of group elements $(x_1, x_2, ..., x_m) \in L^{\times m}$, all corresponding well defined $m-$fold products are equal. A local group is called globally associative if it is associative to every order $n \geq 3$.

2 Homeomorphisms of local top spaces

Theorem 2.1. If H is a regular local top space with finite number of identities which is locally associative then it admits a left invariant frame of rank 0. Conversely if M is a manifold that admits a rank 0 frame, then it can be endowed with the structure of a right regular, locally associative local top space having the given frame as left invariant Lie algebra elements.

Proof. Let H be a locally associative, regular local top space. Given a tangent vector $v_{e(t)} \in TM_{e(t)}$, for $t \in H$, we define $v_x = dl_x v_{e(t)}$. By associativity v is left invariant. Hence $dl_y(v_x) = v(\mu(x, y))$, for any $(x, y) \in U$. By right regularity the set of left invariant vector fields defines a global frame on H.

Conversely if M is a manifold that admits a rank 0 frame, then it can be endowed with the structure of a right regular, locally associative local lie group having the given frame as left invariant Lie algebra elements[3]. Since local lie groups are local top space too the theorem is proved.

Remark 2.1. Any manifold admitting a global frame $V = \{v_1, ..., v_n\}$ also admits a global coframe $\theta = \{\theta_1, ..., \theta_m\}$ which is defined by $< \theta_i, v_j >= \delta^i_j$ at each $x \in M$. The dual structure equations for the coframe are $d\theta_k = \sum C^k_{ij} \theta_i \wedge \theta_j$. The structure functions are the same as structure functions for V. The dual coframe to a left invariant frame on a local top space is known Maurer-Cartan coframe and its structure functions are called Maurer-Cartan equations.

Definition 2.2.[3] A subset $U \subset L$ of a local Lie group L is said to generate L if $L = \bigcup U^{(n)}$. Here

$U^{(n)} \subset L$ denotes the subset consisting of all group elements $x \in L$ which can be written as a well defined n-fold product of elements $x_1, ..., x_n \in U$.

Definition 2.3.[3] A local Lie group L is connected if

- L is a connected manifold,
- The domain of definition of multtiplication and inversion maps are also connected,
- if U is any neighborhood of identity, then U generates L.

Definition 2.4.[3] A local top space H is connected if H be a union of connected local Lie groups.

Theorem.[3] Suppose L and M are connected m-dimensional local Lie groups, and θ, η denote their respective right invariant Maurer-Cartan coframes. If a map $\phi : L \to M$ satisfies $\phi^*(\eta) = \theta$ and and $\phi(e) = \widetilde{e}$, then ϕ defines a local group homeomorphism from L onto its image.

The following theorem is an immediate consequence of pervious theorem.

Theorem 2.2. Suppose that H is a connected local top spaces with finite number of identities, L is a connected local Lie group and θ, η denote their left invariant Maurer-Cartan coframe. If a map $\phi : H \to L$ satisfies $\phi^*(\eta) = \theta$ and $\phi(e(t)) = e$, then ϕ defines a local top space homeomorphism.

Proof. The connected local top space H with the set of identities $\{e_1, ..., e_n\}$ is union of connected local Lie groups $L_1, L_2, ..., L_n$ [1]. Now $\phi_i = \phi|L_1 : L_1 \to L$, $1 \leq i \leq n$, has the properties of the above theorem and consequently is a local homeomorphism. Hence $\phi : H \to L$ is a local homeomorphism too.

Theorem 2.3. Let H be a local top space with finite number of identities. Then there is a covering top space $\widetilde{H}$ of H which is also a local covering of an open subset M of a global Lie group G.

Proof. Since H is union of local Lie groups $L_1, ..., L_n$ there are local covering groups $\phi_i : \overline{L_i} \to L_i$. In addition L_i's and consequently $\overline{L_i}$'s are diffeomorphic. Hence $\overline{L_i}$ are also local covering of M, where $M \subset G$ is an open subset of a global Lie group G [3]. Let $\widetilde{H} = \cup \overline{L_i}$. One can give $\widetilde{H}$ a top space construction such that the map $\phi : \widetilde{H} \to H$, $\phi|\overline{L_i} = \phi_i$, be a local covering. Hence $\widetilde{H}$ is a local covering of M too.

References

[1] N. Ebrahimi, A generalization of Lie theory, Germany: Lap lambert Academic Publishing, 2012, p 43.

[2] M.R. Molaei, "Top spaces," J. Interdiscip. Math. 7(2), pp. 173-181, 2004.

[3] P. J. Olvar, "Non associative local lie groups," presented at the Summer Conference on Topology and its Applications, Howard University, Washington DC, USA, 2003.

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Essential ideals of $C_c(X)$

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Abstract: Let $C_c(X) = \{f \in C(X) : |f(X)| \leq \aleph_0\}$ and $C_F(X)$ be the socle of $C(X)$. Similar to $C(X)$ it is shown that uniform ideals and minimal ideals in $C_c(X)$ coincide. Essential ideals in $C_c(X)$ via a topological property are characterized. We define essential z_c-filter and it is shown that essential z_c-filters behave like z_c-ultrafilters and prime z_c-filters. It is shown that X is a CP-space if and only if every essential ideal in $C_c(X)$ is a z_c-ideal. $C_c(X)$ enjoys most of the important properties of $C(X)$. We observe that if X is an almost discrete space, then $C_F(X)$ is an essential ideal in $C_c(X)$. For essentiality of $C_F(X)$ in some subrings of $C(X)$ the cardinality of $I(X)$ is important. In particular, if $|I(X)| < \infty$, then $C_F(X)$ is not essential in any subring of $C(X)$.

Keywords: essential, zero-dimensional, z_c-ideal, socle, almost discrete space.

1 INTRODUCTION

In [5], the subalgebra $C_c(X)$ of $C(X)$ consisting of functions with countable image are introduced and studied. It turns out that $C_c(X)$, although not isomorphic to any $C(Y)$ in general, it enjoys most of the important properties of $C(X)$ (note, some of these properties do not hold for $C^*(X)$ unless X is finite). The socle of $C(X)$ which is the sum of all minimal ideals of $C(X)$ is characterized topologycally in [6, Proposition 3.3]. In fact, $C_F(X) = \{f \in C(X) : |f(X)| < \infty\}$, and it is a useful object in the context of $C(X)$. In [5], topological spaces in which points and closed sets are separated by elements in $C_c(X)$ are called c-completely regular space and it is observed that these spaces coincide with zero-dimensional ones. Consequently for any topological space X there exists a zero-dimensional space Y which is a continuous image of X and $C_c(X) \cong C_c(Y)$, see [5]. We recall that a nonzero ideal E in a commutative ring R is called essential if it intersects every nonzero ideal nontrivially. An ideal A in $C_c(X)$ is said to be uniform if any two nonzero ideal contained in A intersects nontrivially. Let I be an ideal in $C_c(X)$, then $Z_c[I] = \{Z(f) : f \in I\}$ and $Z_c(X) = \{Z(f) : f \in C_c(X)\}$. If $Z_c^{-1}[Z_c[I]] = I$, then I is called a z_c-ideal. If $F \subseteq Z_c(X)$ be a z-filter, then it is called z_c-filter. The set of uniform ideals in $C_c(X)$ and the set of minimal ideals in $C_c(X)$ coincide similar to $C(X)$. We characterize essential ideals in $C_c(X)$ via a topological property.

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We define essential z_c-filter and it is shown that essential prime z_c-filter behave like z_c-ultrafilter and prime z_c-filter. A space X is called a CP-space if $C_c(X)$ is regular. X is a CP-space if and only if each ideal in $C_c(X)$ is a z_c-ideal. Every P-space is a CP-space and a characterization of CP-spaces similar to one for P-spaces is given in [5]. In zero-dimensional spaces, P-spaces and CP-spaces are the same. We prove that X is a CP-space if and only if every essential ideal in $C_c(X)$ is a z_c-ideal. If X is an almost discrete space, then $C_F(X)$ is an essential ideal in $C_c(X)$. For essentiality $C_F(X)$ in some subrings of $C(X)$ the cardinality of $I(X)$ is important. If $|I(X)| < \infty$, then $C_F(X)$ is not essential in any subring of $C(X)$. All topological spaces that appear are assumed to be infinite completely regular Hausdorff unless the contrary is stated explicity and for undefined terms and notations the reader is referred to [4], [5].

2 Uniform ideals of $C_c(X)$

Theorem 2.1. *Let X be a zero-dimensional space then I is a nonzero minimal ideal of $C_c(X)$ if and only if $|Z_c[I]| = 2$ and for every $0 \neq f \in I$, $|X \backslash Z_c(f)| = 1$ and $X \backslash Z_c(f)$ is closed.*

Corollary 2.2. *Every minimal ideal of $C_c(X)$ is a z_c-ideal.*

The set of uniform ideals in $C_c(X)$ and the set of minimal ideals via $C(X)$ coincide.

Theorem 2.3. *If X is a zero-dimensional space and A is an ideal in $C_c(X)$, then the following are equivalent.*
(i) *A is a uniform ideal in $C_c(X)$.*
(ii) *For any two nonzero elements $f, g \in A$, $fg \neq 0$.*
(iii) *A is a minimal ideal in $C_c(X)$.*

3 Essential ideals in $C_c(X)$

Theorem 3.1. *If X is a zero-dimensional space and E is a nonzero ideal in $C_c(X)$, then the following are equivalent.*
(i) *E intersects every nonzero z_c-ideal in $C_c(X)$ nontrivially.*
(ii) *E is an essential ideal in $C_c(X)$.*
(iii) *$Ann(E) = (0)$.*
(iv) *$\cap Z_c[E]$ is a nowhere dense subset of X.*

proposition 3.2. *If X is a zero-dimensional space, then*
(i) *The Principal ideal $fC_c(X)$ is an essential ideal if and only if $Z_c(f)$ is a nowhere dense subset of X.*
(ii) *Every free ideal I of $C_c(X)$ is an essential ideal.*
(iii) *If x is a nonisolated point in X then the ideals O^c_x and M^c_x are essential ideals.*

Theorem 3.3. *Every pseudoprime ideal in $C_c(X)$ is either an essential ideal or an isolated maximal ideal which is at the same time a minimal prime ideal.*

4 Essential z_c-filter

Next we give a natural definition of an essential z_c-filter.

Definition 4.1. *A z_c-filter F in a space X is called an essential z_c-filter whenever $F \cap F' \neq \{X\}$ for every nontrivial z_c-filter F'.*

The following result shows that the essential z_c-filters behave like the z_c-ultrafilters and prime z_c-filters.

Theorem 4.2. (i) *If E is an essential ideal in $C(X)$, then $Z_c[E]$ is an essential z_c-filter.*
(ii) *If F is an essential z_c-filter, then $Z_c^{-1}[F] = \{f \in C_c(X) : Z_c(f) \in F\}$ is an essential ideal in $C_c(X)$.*

Theorem 4.3. *X is a CP-space if and only if every essential ideal in $C_c(X)$ is a z_c-ideal.*

proposition 4.4. *Every proper ideal of $C_c(X)$ is nonessential if and only if every z_c-ideal of $C_c(X)$ is nonessential.*

5 On essentiality of $C_F(X)$

Theorem 5.1. *Let X be an almost discrete space (i.e., the set of isolated points of X is dense). Then $C_F(X)$ is an essential ideal in any subring T of $C(X)$ which contains $C_F(X)$.*

Corollary 5.2. *If X is an almost discrete space, then $C_F(X)$ is an essential ideal in $C_c(X)$.*

Corollary 5.3. *If X is an almost discrete space, then for every $f \in C_c(X)$ there exists $g \in C_c(X)$ such that $fg \in C_F(X)$.*

The following results show that for essentiality $C_F(X)$ in some subrings of $C(X)$ the cardinality of $I(X)$ is important.

proposition 5.4. *If $|I(X)| \geq \aleph_0$, then $C_F(X)$ is an essential ideal in $R + C_F(X)$.*

Proof. Let $g = r + f$, $0 \neq r \in R$, and $0 \neq f \in C_F(X)$. We note that $C_F(X) = \sum \oplus e_i C(X)$, where for every i, e_i is an idempotent in $C(X)$. Since $f \in C_F(X)$, $f = e_1 f_1 + \ldots + e_n f_n$. Therefore for every $j \neq 1, \ldots, n$, $fe_j = 0$. Hence $ge_j = re_j \in C_F(X) \cap (g)$, and if $r = 0$, then $(g) = (f) \subseteq C_F(X)$. □

proposition 5.5. *Let $|I(X)| < \infty$, then $C_F(X)$ is not essential in any subring of $C(X)$.*

Proof. Let $I(X) = \{x_1, x_2, \ldots, x_n\}$ and

$$e_i(x) = \begin{cases} 1 & , x = x_i \\ 0 & , x \neq x_i \end{cases}$$

$C_F(X) = (e_1, \ldots, e_n) = eC(X)$, where

$$e(x) = \begin{cases} 1 & , x \in \{x_1, \ldots, x_n\} \\ 0 & , x \in X \backslash \{x_1, \ldots, x_n\} \end{cases}$$

is an idempotent and $C(X) = eC(X) \oplus (1-e)C(X)$. Now, if T is a subring of $C(X)$ contains $C_F(X)$, we have $T = (eC(X)(1-e)C(X)) \cap T$. But $T_1 = (1-e)C(X) \cap T$ is an ideal of T such that $C_F(X) \cap T_1 = \{0\}$. □

References

[1] F. Azarpanah, "Essential ideal in $C(X)$," Period. Math. Hungar. 31 (2) (1995), 105-112.

[2] R. Engelking, General topology, Berlin. Heldermann, 1980.

[3] L. Gillman, "Convex and Pseudo prime ideals in C(X)," General topology and its applications, Proceedings of the 1988 Northeast Conference, New York (1990), 87-95.

[4] L. Gillman and M. Jerison, Rings of continuous functions, Springer, 1967.

[5] M. Ghadermazi, O.A.S. Karamzadeh, and M. namdari, "On functionally countable subalgebra of $C(X)$," Rocky Mountain J.Math. 2011, 29, 1153-1164.

[6] O.A.S. Karamzadeh, and M. Rostami, "On the intrinsic topology and some relateds ideals of $C(X)$," Proc. Amer. Math. Soc. 93(1985), 179-184.

[7] W. Dietrich, "On the ideal structure of $C(X)$," Trans. Math. Soc. 152 (1970), 61-77. MR 42. 850.

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Classification of real analytic generic CR-manifolds of dimensions ≤ 5

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Abstract: All real analytic generic CR-manifolds of real dimensions up to five are classified into six general CR-classes denoted by I, II, $\mathsf{III}_1, \mathsf{III}_2$, $\mathsf{IV}_1, \mathsf{IV}_2$. This classification is done in terms of the complex tangent planes associated to the under consideration CR-manifolds. Moreover, general expressions of the determining equations are specified for each class.
Keywords: generic CR-manifolds, equivalence problem.

1 INTRODUCTION

For an N-dimensional complex space $\mathbf{C}^N$ with coordinates $z_k := x_k + i\, y_k$, $k = 1, \ldots, N$, the *complex structure* is defined on the tangent space $T\mathbf{C}^N$ by $J(\frac{\partial}{\partial x_k}) := \frac{\partial}{\partial x_k}$ and $J(\frac{\partial}{\partial y_k}) := -\frac{\partial}{\partial x_k}$. For an arbitrary connected submanifold M of $\mathbf{C}^N$ and for each point $p \in M$, the *complex tangent plane* of M is defined as $T_p^c M := T_p M \cap J(T_p M)$. This bundle can be divided into its conjugated holomorphic and antiholomorphic subbundles $T^{1,0}M$ and $T^{0,1}M$ as $T^c M = T^{1,0}M \oplus T^{0,1}M$. Such submanifold is called *Cauchy-Riemann* (CR for short) whenever the dimension of this space is constant as p varies in M. It also is called *generic* when $T_p M + J(T_p M) = T_p \mathbf{C}^N$. As is known [8, 4], every real analytic generic CR-manifold $M \subset \mathbf{C}^N$ can be represented locally in some coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_c)$ with $w = u + i\, v$ and with $N = n + c$ as the graph of some c real functions:

$$\left[\begin{array}{l} v_1 := \varphi(z, \overline{z}, u), \\ \vdots \\ v_c := \varphi(z, \overline{z}, u). \end{array}\right.$$

In this case, the integers n and c are in fact the CR-dimension and CR-codimension of M and the real dimension of M is $\dim_{\mathbf{R}} M = 2n + c$.

2 CLASSIFICATION

Our aim in this paper is to classify all real analytic generic CR-manifolds of real dimensions ≤ 5. At first, one should notice that the two cases $n = 0$ and $c = 0$ are not interesting in CR-geometry, for:

$$M \cong \mathbf{R}^c,$$
$$M \cong \mathbf{C}^n,$$

respectively. Hence, one assumes $n, c \geq 1$. A plain inspection shows that there are in fact four possibilities for n and c so that the corresponding CR-manifolds being of real dimensions ≤ 5:

$$\boxed{\begin{array}{lcl} 2n + c = 3 & \Longrightarrow & \{n = 1, \quad c = 1, \\ 2n + c = 4 & \Longrightarrow & \{n = 1, \quad c = 2, \\ 2n + c = 5 & \Longrightarrow & \left\{\begin{array}{ll} n = 1, & c = 3, \\ n = 2, & c = 1. \end{array}\right. \end{array}}$$

In order to distinguish these cases, dimensions must be emphasized:

$$M^3 \subset \mathbf{C}^2, \quad M^4 \subset \mathbf{C}^3, \quad M^5 \subset \begin{array}{c} \mathbf{C}^4, \\ \mathbf{C}^3 \end{array}.$$

Before proceeding to classify these CR-manifolds, we need the following result from [4].

Proposition 2.1. *On a CR-generic submanifold $M^{2n+c} \subset \mathbf{C}^{n+c}$ of smoothness $\mathcal{C}^\kappa$ ($\kappa \geq 1$) which is locally represented in coordinates:*

$$\big(z_1, \dots, z_n, \, w_1, \dots, w_c\big)$$

with $w = u + i\,v$ as the graph of c real valued functions:

$$v_1 = \varphi_1\big(z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n, u_1, \dots, u_c\big),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$v_c = \varphi_c\big(z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n, u_1, \dots, u_c\big),$$

a local frame for $T^{1,0}M$:

$$\{\mathcal{L}_1, \dots, \mathcal{L}_n\}$$

is constituted of the n vector fields:

$$\mathcal{L}_1 = \frac{\partial}{\partial z_1} + \mathsf{A}_1^1(x_\bullet, y_\bullet, u_\bullet)\,\frac{\partial}{\partial w_1} + \cdots + \mathsf{A}_1^c(x_\bullet, y_\bullet, u_\bullet)\,\frac{\partial}{\partial w_c}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathcal{L}_n = \frac{\partial}{\partial z_n} + \mathsf{A}_n^1(x_\bullet, y_\bullet, u_\bullet)\,\frac{\partial}{\partial w_1} + \cdots + \mathsf{A}_n^c(x_\bullet, y_\bullet, u_\bullet)\,\frac{\partial}{\partial w_c},$$

whose coefficient functions are given, for $i = 1, \dots, n$, explicitly by:

$$\mathsf{A}_i^1 = \frac{\begin{vmatrix} -2\,\varphi_{1,z_i} & \varphi_{1,u_2} & \cdots & \varphi_{1,u_c} \\ -2\,\varphi_{2,z_i} & i + \varphi_{2,u_2} & \cdots & \varphi_{2,u_c} \\ \vdots & \vdots & \ddots & \vdots \\ -2\,\varphi_{c,z_i} & \varphi_{c,u_2} & \cdots & i + \varphi_{c,u_c} \end{vmatrix}}{\begin{vmatrix} i + \varphi_{1,u_1} & \cdots & \varphi_{1,u_c} \\ \varphi_{2,u_1} & \cdots & \varphi_{2,u_c} \\ \vdots & \ddots & \vdots \\ \varphi_{c,u_1} & \cdots & i + \varphi_{c,u_c} \end{vmatrix}}, \ldots\ldots,$$

$$\mathsf{A}_i^c = \frac{\begin{vmatrix} i + \varphi_{1,u_1} & \cdots & -2\,\varphi_{1,z_i} \\ \varphi_{2,u_1} & \cdots & -2\,\varphi_{2,z_i} \\ \vdots & \ddots & \vdots \\ \varphi_{c,u_1} & \cdots & -2\,\varphi_{c,z_i} \end{vmatrix}}{\begin{vmatrix} i + \varphi_{1,u_1} & \cdots & \varphi_{1,u_c} \\ \varphi_{2,u_1} & \cdots & \varphi_{2,u_c} \\ \vdots & \ddots & \vdots \\ \varphi_{c,u_1} & \cdots & i + \varphi_{c,u_c} \end{vmatrix}}. \quad \square$$

This proposition suggests one to proceed the desired classification according to the structure of Lie algebra generated by the associated local frame of $T^cM = T^{1,0}M \oplus T^{0,1}M$. By this general strategy, it is found in [4] and [3] six (holomorphically) non-equivalence classes of real analytic generic CR-manifolds of dimensions ≤ 5, denoted by $\mathsf{I}, \mathsf{II}, \mathsf{III}_1, \mathsf{III}_2, \mathsf{IV}_1, \mathsf{IV}_2$.

2.1 General class I

The first class I comprises all 3-dimensional CR-manifolds $M^3 \subset \mathbf{C}^2$ of the equal CR-dimension and CR-codimension 1 such that for any local vector field generator $\mathcal{L}$ of $T^{1,0}M^3$ the set $\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]\}$ constitutes a frame for the complexified bundle $\mathbf{C} \otimes_{\mathbf{R}} TM^3$.

Proposition 2.2. *In coordinates $(z, w = u + i\,v) \in \mathbf{C}^2$, every real analytic generic CR-manifold belonging to the general class I can be represented as the graph of a single certain polynomial of the form:*

$$\underline{(\mathsf{I})}: \qquad v = z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z, \overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u).$$

Here by $\mathrm{O}_t(\mathsf{x})$, we mean monomials of degrees $\geq t$ in terms of the variable(s) x.

2.2 General class II

This class, comprises all real analytic generic CR-manifolds $M^4 \subset \mathbf{C}^3$ of real dimension 4, of CR-dimension 1 and of CR-codimension 2, in which for any local generator $\mathcal{L}$ of $T^{1,0}M^4$ the set $\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}], [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]\}$ constitutes a frame for the complexified bundle $\mathbf{C} \otimes_{\mathbf{R}} TM^4$.

Proposition 2.3. *In coordinates $(z, w_1 = u_1 + i\,v_1, w_2 = u_2 + i\,v_2) \in \mathbf{C}^3$, every real analytic generic CR-manifold belonging to the class II can be represented as the graph of two certain polynomials of the form:*

$$\underline{(\mathsf{II})}: \qquad \begin{aligned} v_1 &= z\overline{z} + z\overline{z}\,\mathrm{O}_2\big(z, \overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2), \\ v_2 &= z^2\overline{z} + z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z, \overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2). \end{aligned}$$

2.3 General class III_1

This third general class comprises all 5-dimensional CR-manifolds $M^5 \subset \mathbf{C}^4$ of CR-dimension 1 and CR-codimension 3 such that for any local generator $\mathcal{L}$ of $T^{1,0}M^5$, the set $\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}], [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]], [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]]\}$ constitutes a frame for the complexified bundle $\mathbf{C} \otimes_{\mathbf{R}} TM^5$.

Proposition 2.4. *In coordinates $(z, w_1 = u_1 + i\,v_1, w_2 = u_2 + i\,v_2, w_3 = u_3 + i\,v_3) \in \mathbf{C}^4$, every real analytic generic CR-manifold belonging to the class III_1 can be represented as the graph of three certain polynomials of the form:*

$$
(\mathsf{III})_1: \quad
\begin{aligned}
v_1 &= z\overline{z} + z\overline{z}\,\mathrm{O}_2(z, \overline{z}) + z\overline{z}\,\mathrm{O}_1(u_1) \\
&\qquad + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3), \\
v_2 &= z^2\overline{z} + z\overline{z}^2, + z\overline{z}\,\mathrm{O}_2(z, \overline{z}) + z\overline{z}\,\mathrm{O}_1(u_1) \\
&\qquad + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3), \\
v_3 &= i\,(z^2\overline{z} - z\overline{z}^2) + z\overline{z}\,\mathrm{O}_2(z, \overline{z}) \\
&\qquad + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3).
\end{aligned}
$$

2.4 General class III_2

Similar to III_1, this class includes 5-dimensional CR-manifolds $M^5 \subset \mathbf{C}^4$ of CR-dimension 1 and CR-codimension 3, too. The difference is that in this case and for any local generator $\mathcal{L}$ of $T^{1,0}M^5$, one has at every point:

$$
\begin{aligned}
\mathbf{3} &= \mathsf{rank}_{\mathbf{C}}\big(\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]\}\big), \\
\mathbf{4} &= \mathsf{rank}_{\mathbf{C}}\big(\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}], [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]\}\big), \\
\mathbf{4} &= \mathsf{rank}_{\mathbf{C}}\big(\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}], [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]], [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]]\}\big), \\
\mathbf{5} &= \mathsf{rank}_{\mathbf{C}}\big(\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}], [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]], [\mathcal{L}, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]]\}\big).
\end{aligned}
$$

More precisely and in contrary to the class III_1, here the complexified bundle $\mathbf{C} \otimes TM^5$ can not be generated by the set $\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}], [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]], [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]]\}$ since the rank of this subbundle is $4 \leq \dim_{\mathbf{R}} M^5 = 5$. Hence, one needs to replace the length 3 Lie bracket $[\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]]$ by the length 4 bracket $[\mathcal{L}, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]]$ and, because of its rank, the new set $\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}], [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]], [\mathcal{L}, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]]\}$ constitutes a frame for $\mathbf{C} \otimes TM^5$.

Proposition 2.5. *In coordinates $(z, w_1 = u_1 + i\,v_1, w_2 = u_2 + i\,v_2, w_3 = u_3 + i\,v_3) \in \mathbf{C}^4$, every real analytic generic CR-manifold belonging to the class III_2 can be represented as the graph of three certain polynomials of the form:*

$$
(\mathsf{III})_2: \quad
\begin{aligned}
v_1 &= z\overline{z} + c_1\,z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_3(z, \overline{z}) + z\overline{z}\,u_1\,\mathrm{O}_1(z, \overline{z}, u_1) \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1(z, \overline{z}, u_1, u_2) + z\overline{z}\,u_3\,\mathrm{O}_1(z, \overline{z}, u_1, u_2, u_3), \\
v_2 &= z^2\overline{z} + z\overline{z}^2 + z\overline{z}\,\mathrm{O}_3(z, \overline{z}) + z\overline{z}\,u_1\,\mathrm{O}_1(z, \overline{z}, u_1) \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1(z, \overline{z}, u_1, u_2) + z\overline{z}\,u_3\,\mathrm{O}_1(z, \overline{z}, u_1, u_2, u_3), \\
v_3 &= 2\,z^3\overline{z} + 2\,z\overline{z}^3 + 3\,z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_3(z, \overline{z}) \\
&\qquad + z\overline{z}\,u_1\,\mathrm{O}_1(z, \overline{z}, u_1) + z\overline{z}\,u_2\,\mathrm{O}_1(z, \overline{z}, u_1, u_2) \\
&\qquad + z\overline{z}\,u_3\,\mathrm{O}_1(z, \overline{z}, u_1, u_2, u_3).
\end{aligned}
$$

2.5 General class IV_1

This is the first class that comprises CR-manifolds of CR-dimension 2. More precisely, every 5-dimensional CR-manifold M^5 of $\mathbf{C}^3$ in which for any two local generator fields $\mathcal{L}_1, \mathcal{L}_2$ of $T^{0,1}M^5$, the set $\{\mathcal{L}_1, \mathcal{L}_2, \overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, [\mathcal{L}_1, \overline{\mathcal{L}}_1]\}$ constitutes a frame for the complexified bundle $\mathbf{C} \otimes_{\mathbf{R}} TM^5$ belongs to this class.

Proposition 2.6. *In coordinates $(z_1, z_2, w = u + i\,v) \in \mathbf{C}^3$, every real analytic generic CR-manifold belonging to the class IV_1 can be represented as the graph of a single polynomial of the form:*

$$
(\mathsf{IV})_1: \quad v = z_1\overline{z}_1 \pm z_2\overline{z}_2 + \mathrm{O}_3(z_1, z_2, \overline{z}_1, \overline{z}_2, u),
$$

with remainders satisfying:

$$
0 \equiv \mathrm{O}_3(0, 0, \overline{z}_1, \overline{z}_2, u) \equiv \mathrm{O}_3(z_1, z_2, 0, 0, u).
$$

2.6 General class IV_2

Similar to IV_1, this class includes 5-dimensional real analytic generic CR-manifolds $M^5 \subset \mathbf{C}^3$ of CR-dimension 2 and codimension 1 such that for any two local generator fields $\mathcal{L}_1, \mathcal{L}_2$ of $T^{0,1}M^5$, the set $\{\mathcal{L}_1, \mathcal{L}_2, \overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, [\mathcal{L}_1, \overline{\mathcal{L}}_1]\}$ constitutes a frame for $\mathbf{C} \otimes_{\mathbf{R}} TM^5$. But in contrary, in this case we assume that the Levi form $\mathsf{Levi\text{-}Form}^M(p)$ is of constant rank and such that lastly, the Freeman form $\mathsf{Freeman\text{-}Form}^M(p)$ is nondegenerate at every point (see [8] for the relevant definitions).

Proposition 2.7. *In coordinates* $(z_1, z_2, w = u + i\,v) \in \mathbf{C}^3$*, every real analytic generic CR-manifold belonging to the class* IV_2 *can be represented as the graph of a single polynomial of the form:*

$$(\mathsf{IV})_2: \quad \begin{aligned} v &= z_1\overline{z}_1 + \tfrac{1}{2}\,z_1z_1\overline{z}_2 + \tfrac{1}{2}\,z_2\overline{z}_1\overline{z}_1 \\ &\quad + \mathrm{O}_4(z_1, z_2, \overline{z}_1, \overline{z}_2) + u\,\mathrm{O}_2(z_1, z_2, \overline{z}_1, \overline{z}_2, u), \end{aligned}$$

with remainders both satisfying:

$$0 \equiv \mathrm{O}\big(0, 0, \overline{z}_1, \overline{z}_2, u\big) \equiv \mathrm{O}\big(z_1, z_2, 0, 0, u\big).$$

3 THE EQUIVALENCE PROBLEM

Among the recent century, the problem of equivalence of CR-manifolds of various dimensions had been one of the most interesting investigated questions. In 1907, Henri Poincaré initiated the problem of equivalence of real hypersurfaces of CR-dimension and codimension 1, those constitute the first general class I. This problem solved later on by Élie Cartan [2] in 1932, whom initiated also a powerful algorithm to answer similar questions in CR-geometry, in differential geometry, in mathematical physics, in differential equations and many others. Afterwards, the problem of equivalences in general class I and also its associated Cartan geometry investigated several times for example in [6, 5]. The equivalence problem of the second class II has been recently considered in [1], where the authors also investigated the associated Cartan geometry in the framework of Tanaka theory. But *undoubtedly*, the most complicated problem is the equivalence of the elements of the general class III_1. This problem has been solved very recently by the author jointed by Joël Merker in the long memoir [7]. Finally, the equivalence problem of the next three classes $\mathsf{III}_2, \mathsf{IV}_1, \mathsf{IV}_2$ is in progress by the author jointed with Joël Merker and his Ph.D student Samuel Pocchiola and will be presented very soon in the near future.

To the best of our knowledge, there is no any classification of CR-manifolds of real dimensions bigger than 5 and also it is the maximum dimension that the equivalence of CR-manifolds is considered. In other words, classification of CR-manifolds of higher dimensions and the solution of their equivalence problems are *still open*.

References

[1] V. K. Beloshapka, V. Ezhov, and G. Schmalz, Canonical Cartan connection and holomorphic invariants on Engel CR manifolds, Russian J. Mathematical Physics **14** (2007), no. 2, 121–133.

[2] É. Cartan, Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, II, Ann. Scuola Norm. Sup. Pisa **1** (1932), 333–354.

[3] J. Merker, "Equivalences of 5-dimensional CR manifolds, III: Six models and (very) elementary normalizations," 54 pages, arxiv.org/abs/1311.7522/

[4] J. Merker, S. Pocchiola, and M. Sabzevari, "Equivalences of 5-dimensional CR manifolds, II: General classes I, II, III_1, III_2, IV_1, IV_2," 5 figures, 95 pages, arxiv.org/abs/1311.5669/.

[5] J. Merker, and M. Sabzevari, Explicit expression of Cartan's connections for Levi-nondegenerate 3-manifolds in complex surfaces, and identification of the Heisenberg sphere, Cent. Eur. J. Math., **10**(5), (2012), 1801–1835.

[6] M. Sabzevari, and J. Merker, "Cartan equivalence problem for Levi-nondegenerate real hypersurfaces $M^3 \subset \mathbf{C}^2$," Izvest.: Math., to appear.

[7] M. Sabzevari, and J. Merker, "Cartan equivalence problem for general 5-dimensional CR-manifolds in $\mathbf{C}^4$," arXiv:1401.4297v1, 172 pages.

Book:

[8] A. Boggess, "CR manifolds and the tangential Cauchy-Riemann complex," Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1991, xviii+364 pp.

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The axiom of spheres in Finsler geometry

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Abstract: Here, an axiom of r-spheres in Finsler geometry is proposed and it is proved that if a Finslerian manifold satisfies the axiom of r-spheres then M has constant sectional curvature in the Cartan connection.
Keywords: Finsler space, totally umbilical submanifold, Codazzi equations.

1 INTRODUCTION

In Riemannian geometry, E. Cartan defined the axiom of r-planes as follows. A Riemannian manifold M of dimension $n \geq 3$ satisfies the axiom of r-palnes, where r is a fixed integer $2 \leq r < n$, if for each point p of M and any r-dimensional subspace S of the tangent space T_pM there exists an r-dimensional totally geodesic submanifold V containing p such that $T_pV = S$. He proved that if M satisfies the axiom of r-planes for some r, then M has constant sectional curvature, cf., [6]. The axiom of r-spheres in Riemannian geometry was proposed by Leung and Nomizu as follows. For each point p of M and any r-dimensional subspace S of T_pM, there exists an r-dimensional umbilical submanifold V with parallel mean curvature vector field such that $p \in V$ and $T_pV = S$. They proved that if a Riemannian manifold M of dimension $n \geq 3$ satisfies the axiom of r-spheres for some r, $2 \leq r < n$, then M has constant sectional curvature, cf., [2]. It is shown that if one drop in the axiom of spheres the requirement that V has parallel mean curvature vector field, then this weaker axiom for $n \geq 4$ and $r = n - 1$ implies that M is confopmally flat. One could extend this result to the case $3 \leq r < n$. In Finsler geometry, Akbar-Zadeh defined the axiom of 2-planes as follows. A Finslerian manifold M of dimension $n \geq 3$ satisfies the axiom of 2-planes if for each point $p \in M$ and every subspace E_2 of dimension two of T_pM there exists a totally geodesic surface S passing through p such that $T_pS = E_2$. He proved that every Finsler manifold satisfying axiom of 2-planes is of constant sectional curvature in the connection of Berwald, cf., [3], page 182. In this paper, we propose the axiom of spheres in Finsler spaces and prove that if a Finsler manifold M of dimension $n \geq 3$ satisfies the axiom of spheres then it has constant sectional curvature in the Cartan connection.

2 PRELEMINARIES

Let M be a real n-dimensional manifold of class C^∞. We denote by TM the tangent bundle of tangent vectors, by $p : TM_0 \longrightarrow M$ the fiber bundle of non-zero tangent vectors and by $p^*TM \longrightarrow TM_0$

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the pulled-back tangent bundle. Let (x, U) be a local chart on M and (x^i, y^i) the induced local coordinates on $p^{-1}(U)$. A *Finsler structure* on M is a function $F : TM \longrightarrow [0, \infty)$, with the following properties:(i) F is differentiable C^∞ on TM_0; (ii) F is positively homogeneous of degree one in y, that is, $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$; (iii) The Finsler metric tensor g defined by the Hessian matrix of F^2, $(g_{ij}) = (\frac{1}{2}[\frac{\partial^2}{\partial y^i \partial y^j}F^2])$, is positive definite on TM_0. A *Finsler manifold* is a pair (M, F) consisting of a differentiable manifold M and a Finsler structure F on M. We denote by TTM_0, the tangent bundle of TM_0 and by ρ, the canonical linear mapping $\rho : TTM_0 \longrightarrow p^*TM$, where, $\rho = p_*$. There is the horizontal distribution HTM such that we have the Whitney sum $TTM_0 = HTM \oplus VTM$. This decomposition permits to write a vector field $\hat{X} \in \chi(TM_0)$ into the horizontal and vertical parts in a unique manner, namely $\hat{X} = H\hat{X} + V\hat{X}$. In the sequel, we denote all vector fields on TM_0 by $\hat{X}, \hat{Y}$, etcetera and the corresponding sections of p^*TM by $X = \rho(\hat{X})$, $Y = \rho(\hat{Y})$, respectively, unless otherwise specified, cf., [3].

2.1 Geometry of submanifolds

Let $i : S \longrightarrow M$ be an immersion and S a submanifold of dimension k of the manifold M. We identify any point $x \in S$ by its image $i(x)$ and any tangent vector $X \in T_xS$ by its image $i_*(X)$, where i_* is the linear tangent mapping. Thus T_xS becomes a sub-space of T_xM. Let TS_0 be the fiber bundle of non-zero tangent vectors on S. TS_0 is a sub-vector bundle of TM_0 and the restriction of p to TS_0 is denoted by $q : TS_0 \longrightarrow S$. We denote by $\bar{T}(S) = i^*TM$, the pull back induced vector bundle of TM by i. The Finslerian metric of M induces a Finslerian metric on S where we denote it again by g. At a point $x = qz \in S$, where $z \in TS_0$, the orthogonal complement of $T_{qz}S$ in $\bar{T}_{qz}S$ is denoted by $N_{qz}S$, namely, $\bar{T}_x(S) = T_x(S) \oplus N_{qz}S$, where $T_x(S) \cap N_{qz}S = 0$. We have the following decomposition:

$$q^*\bar{T}S = q^*TS \oplus N, \tag{1}$$

where, N is called the normal fiber bundle. If TTS_0 is the tangent vector bundle to TS_0, we denote by ϱ, the canonical linear mapping $\varrho : TTS_0 \longrightarrow q^{-1}TS$. Let $\hat{X}$ and $\hat{Y}$ be the two vector field on TS_0. For $z \in TS_0$, $(\nabla_{\hat{X}}Y)_z$ belongs to $\bar{T}_{qz}S$. Attending to (1) we have

$$\nabla_{\hat{X}}Y = \bar{\nabla}_{\hat{X}}Y + \alpha(\hat{X}, Y), \quad Y = \varrho(\hat{Y}), \quad X = \varrho(\hat{X}), \tag{2}$$

where ∇ is the covariant derivative of Cartan connection and $\alpha(\hat{X}, Y)$ the second fundamental form of the sub-manifold S. It belongs to N and is bilinear in $\hat{X}$ and Y. It results from (2) that the induced connection $\bar{\nabla}$ is a metric compatible covariant derivative with respect to the induced metric g in the vector bundle $q^*TS \longrightarrow TS_0$.

2.2 Shape operator or Weingarten formula

Let S be an immersed submanifold of (M, F). For any $\hat{X} \in \chi(TS_0)$ and $W \in \Gamma(N)$ we set

$$\nabla_{\hat{X}}W = -A_W\hat{X} + \bar{\nabla}^{\perp}_{\hat{X}}W, \tag{3}$$

where $A_W\hat{X} \in \Gamma(q^*TS)$ and $\bar{\nabla}^{\perp}_{\hat{X}}W \in \Gamma(N)$. It follows that $\bar{\nabla}^{\perp}$ is a linear connection on the normal bundle N, cf., [5]. We also consider the bilinear map

$$A : \Gamma(N) \otimes \Gamma(TTS_0) \longrightarrow \Gamma(q^*TS),$$
$$A(W, \hat{X}) = A_W\hat{X}.$$

For any $W \in \Gamma(N)$, the operator $A_W : \Gamma(TTS_0) \longrightarrow \Gamma(q^*TS)$ is called the *shape operator* or the Weingarten map with respect to W. Finally, (3) is said to be the *Weingarten formula* for the immersion of S in M, cf., [5].

2.3 Totally umbilical submanifolds in Finsler spaces

The *mean curvature* vector field η of the isometric immersion $i : S \longrightarrow M$ is defined by

$$\eta = \frac{1}{n} tr_g \alpha({}^h\hat{X}, Y), \tag{4}$$

where $X, Y \in \Gamma(q^*TS)$ and ${}^h\hat{X}$ is the horizontal lift of X, cf., [1]. We say that the mean curvature vector field η is parallel in all direction if $\bar{\nabla}^{\perp}_{{}^h\hat{X}}\eta = 0$ for all $X \in \Gamma(q^*TS)$.

Definition 2.1. *[1] A submanifold of a Finsler manifold is said to be totally umbilical, or simply umbilical, if it is equally curved in all tangent directions.*

More precisely, let $i : S \longrightarrow M$ be an isometric immersion. Then i is called totally umbilical if there exists a normal vector field $\xi \in N$ along i such that its second fundamental form α with values in the normal bundle satisfies

$$\alpha({}^h\hat{X}, Y) = g(X, Y)\xi, \tag{5}$$

for all $X, Y \in \Gamma(q^*TS)$, where ${}^h\hat{X}$ is the horizontal lift of X.

Remark 2.2. *Let $i : S \longrightarrow M$ be an isometric immersion. If S is totally umbilical then the normal vector field ξ is equal to the mean curvature vector field η.*

2.4 Genaralization of Schur's theorem

Let $G_2(M)$ be the Grassmannian fiber bundle of 2-planes on M. Denote by $\pi^{-1}G_2(M) \longrightarrow SM$ the fiber induced on SM by $\pi : SM \longrightarrow M$, where SM is the unit sphere bundle. Let $P \in \pi^{-1}G_2(M)$ be a 2-plane generated by vectors X and Y linearly independent at $x = \pi z \in M$ where $z \in SM$. By means of the hh-curvature tensor R of the Cartan connection one defines the function $K_2 : \pi^{-1}G_2(M) \longrightarrow \Re$ as follows

$$K_2(z, X, Y) = \frac{g(R(X, Y)Y, X)}{\| X \|^2 \| Y \|^2 - g(X, Y)^2}.$$

K_2 will be called the sectional curvature at $z \in SM$ following the 2-plane $P(X, Y)$ generated by X and Y in the Cartan connection. We have

$$K_2(z, v, X) = K(z, v, X),$$

where v is the canonical section and K is the flag curvature, cf., [3], page 156.

Genaralization of Schur's theorem. $K_2(z, P)$ is independent of 2-plane $P(X, Y)$ (dim $M > 2$) if and only if the curvature tensor R of the Cartan connection is defined by

$$R(X, Y)Z = K[g(Y, Z)X - g(X, Z)Y],$$

where K is a constant and $X, Y, Z \in T_xM$, cf., [3].

3 MAIN RESULTS

We propose the axiom of r-spheres in Finler spaces as follows:

Axiom of r-sphere. Let (M, g) be a Finsler manifold of dimension $n \geq 3$. For each point x of M and any r-dimensional subspace E_r of T_xM, there exists an r-dimensional umbilical submanifold S with parallel mean curvature vector field such that $x \in S$ and $T_xS = E_r$.

We shall prove

Theorem 3.1. *If a Finsler manifold of dimension $n \geq 3$ satisfies the axiom of r-spheres for some r, $2 \leq r < n$, then M has constant sectional curvature in Cartan connection.*

sketch of proof. Suppose (M, g) be a Finsler manifold that satisfies the axiom of r-spheres and consider the Cartan connection on the pulled-back bundle p^*TM. Let X, Y and Z be three orthonormal vectors at $x = pz, z \in TM$ where x is an arbitrary point of M and E_r be an r-dimensional subspace of T_xM containing X and Y and normal to Z. By the axiom there exists an r-dimensional umbilical submanifold S with parallel mean curvature vector field η such that $x \in S$ and $T_xS = E_r$. By means of Codazzi equations in

Finsler spaces we show that for vectors $X, Y \in T_xS$ and $Z \in N_xS$ we have

$$g(R(X,Y)Y,X) = g(R(X,Z)Z,X). \qquad (6)$$

Thus the common value of (6) does not depend on the 2-plane $P(X,Y)$. Therefore by means of generalization of Schur's theorem M has constant sectional curvature.

References

[1] L. M. Abatangelo, "On totally umbilical submanifolds of locally Minkowski manifold", Collect. Math. 43, pp. 151-175 ,(1992).

[2] D. S. Leung, K. Nomizu, "The axiom of spheres in Riemannian geometry", J. Differential Geometry, Vol. 5, no. 3-4, pp. 487-489, (1971).

Books:

[3] H. Akbar-Zadeh, "Initiation to global Finslerian geometry", vol. 68. Elsevier Science, 2006.

[4] D. Bao, S. Chern, Z. Shen, "An introduction to Riemann-Finsler Geometry. Graduate Texts in Mathematics", Vol. 200, Springer, 2000.

[5] A. Bejanco, H. R. Farran, "Geometry of pseudo-Finsler submanifolds", Kluwer Acad. Publ., Dordrecht, 2000.

[6] E. Cartan, "Lecons sur la geometrie des espaces de Riemann", Gauthier-Villars, Paris, 1946.

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Ultrafilters as infinitesimal points in topological space

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Abstract: Nonstandard analysis is very complex, so finding a simple description of infinitesimal points will be useful. In this paper, ultrafilters as infinitesimal points in a topological space will be proposed, and some topological concepts is restated by this tools.
Keywords: The Stone-Čech compactification, Axiom of separation, Nonstandard Analysis, Nonstandard Topology, monad.

1 INTRODUCTION

Nonstandard Analysis initiated by Newton and Leibniz, was accompanied by logical contradictions which advances in mathematical logic in the twentieth century could resolve them. Nonstandard methods can give a special insight in topological matters as they are mainly a new way to look at old things. Using a type-theoretical version of higher-order logic, A. Robinson in 1960 [6], introduced the notion of an enlargement which is a main tool in nonstandard topology. An enlargement is a certain kind of nonstandard model satisfying a sort of saturation property which is closely related to an essential feature of nonstandard methods in topology, i.e., compactness.

The connections between nonstandard extensions and ultrafilters have been repeatedly considered in the literature, starting from the seminal paper by Luxemburg. M. Di Nasso and M. Forti introduced a notion of topological extension of a given set X. The resulting class of topological spaces includes the Stone-Čech compactification βX of the discrete space X, as well as all nonstandard models of X in the sense of nonstandard analysis (when endowed with a natural topology). They gave a simple characterization of nonstandard extensions in purely topological terms, and they established connections with special classes of ultrafilters whose existence is independent of ZFC.

In [4], the ultrafilter semigroup $(0^+, +)$ of the topological semigroup $T = ((0, +\infty), +)$ consists of all nonprincipal ultrafilter on $T = (0, +\infty)$ converging to the 0 has been described. According to [4], in a topological semigroup T, [7] has been presented.

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2 Preliminary

Let Γ be a family of sets that together with A and B contains the intersection $A \cap B$. By a filter in Γ we mean a non-empty subfamily $\mathcal{F} \subseteq \Gamma$ satisfying the following conditions:
(F_1) $\emptyset \notin \mathcal{F}$.
(F_2) If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$.
(F_3) If $A \in \mathcal{F}$ and $A \subseteq B \in \Gamma$, then $B \in \mathcal{F}$.
A filter $\mathcal{F}$ in Γ is a maximal filter or an ultrafilter in Γ, if for every filter $\mathcal{A}$ in Γ that contains $\mathcal{F}$ we have $\mathcal{A} = \mathcal{F}$.

A filter $\mathcal{F}$ on (X, τ) converges to a point $x \in X$ if $\tau_x \subseteq \mathcal{F}$, where

$$\tau_x = \{U \subset X : x \in V \subseteq U \text{ for some } V \in \tau\}$$

is the collection of all neighborhoods of $x \in X$. In this case, the point is called a limit of the filter $\mathcal{F}$ and we write $x \in lim\mathcal{F}$. A point x is called a cluster point of a filter $\mathcal{F}$ if x belongs to the closure of every member of $\mathcal{F}$. Clearly, x is a cluster point of a filter $\mathcal{F}$ if and only if every neighborhood of x intersects all members of $\mathcal{F}$. This implies in particular that every cluster point of an ultrafilter is a limit of this ultrafilter. It is obvious that a subset $A \subseteq X$ is closed in τ if and only if a limit of any filter containing A belongs to A.

Let X_d denote X with discrete topology. Now we describe the Stone-$\check{C}$ech compactification βX_d of X_d. We take the points of βX_d to be the ultrafilters on X_d, identifying the points of X_d with $\widehat{x} = \{A \subseteq X_d : x \in A\}$ and let $\widehat{A} = \{p \in \beta X_d : A \in p\}$ for $A \subseteq X$. The topology of βX_d is defined by stating $\{\widehat{A} : A \subseteq X\}$ as a base for the open sets. Then $cl_{\beta X_d} A = \widehat{A}$ for $A \subseteq X_d$. If $\mathcal{A}$ be a filter then $\overline{\mathcal{A}} = \{p \in \beta X_d : \mathcal{A} \subseteq p\}$ is a closed subset of βX_d. Also, if T be a subset of βX_d, then $\mathcal{A} = \bigcap T$ is a filter and $cl_{\beta X_d} T = \overline{\mathcal{A}}$. For more details see [5] and [8]. If X be a completely regular space, then the Stone-$\check{C}$ech compactification of X has been described by similar way as space of z-ultrafilters, see [2].

If X and Y are any completely regular spaces, then any continuous function $f : X Rightarrow Y$ has a unique continuous extension $f^\beta : \beta X \to \beta Y$.

Lemma 2.1. *Let $f : X_d \to Y_d$ be a function. For each $p \in \beta X_d$,*

$$f^\beta(p) = \{A \subseteq X_d : f^{-1}(A) \in p\}.$$

In particular, if $A \in p$, then $f(A) \in f^\beta(p)$.

Proof : See [5].

Now we review the definition of partition regularity and a theorem that connect it with ultrafilters.

Definition 2.2. *Let R be a nonempty set of subsets of X. R is partition regular if and only if whenever $\mathcal{F}$ is a finite set of $\mathcal{P}(X)$ ($\mathcal{P}(X)$ is the set of all subsets of X) and $\bigcup \mathcal{F} \in R$, there exist $A \in \mathcal{F}$ and $B \in R$ such that $B \subseteq A$.*

Theorem 2.3. *Let $R \subseteq \mathcal{P}(X)$ be a nonempty set and assume $\emptyset \notin R$. Let*

$$R^\uparrow = \{B \in \mathcal{P}(X) : A \subseteq B \text{ for some } A \in R\}.$$

Then (a), (b) and (c) are equivalence.
(a) R is partition regular.
(b) Whenever $\mathcal{A} \subseteq \mathcal{P}(X)$ has the property that every finite nonempty subfamily of $\mathcal{A}$ has an intersection which is in $R^\uparrow$, there is $\mathcal{U} \in \beta G_d$ such that $\mathcal{A} \subseteq \mathcal{U} \subseteq R^\uparrow$.
(c) Whenever $A \in R$, there is $\mathcal{U} \in \beta X_d$ such that $A \in \mathcal{U} \subseteq R^\uparrow$.

Proof. [5, Theorem 3.11].

3 Description of topological concepts by ultrafilters

Let (X, τ) be a topological space. For $x \in X$, with respect to τ on X, we define

$$x^* = \{p \in \beta X_d : x \in \bigcap_{A \in p} cl_X A\}.$$

In fact, x^* is the collection of all ultrafilters converge to x. It is obvious that $\widehat{x} \in x^*$. We say $p \in x^*$

is a near point to x. We define $B(X) = \bigcup_{x\in X} x^*$ and $\infty^* = \beta X_d - B(X)$. $p \in B(X)$ is called bounded ultrafilter and $p \in \infty^*$ is called unbounded ultrafilter. For $F \subseteq X$ define $F^* = \{p \in \beta X_d : p \subseteq R_F$, where $R_F = \{A \subseteq X : cl_X(A) \cap F \neq \emptyset\}$. It is obvious that $x^* \subseteq F^*$, for each $x \in F$.

Theorem 3.1. *Let (X, τ) be a Hausdorff topological space.*
a) $p \in \infty^*$ if and only if for each $x \in X$ there exists $A \in p$ such that $x \notin cl_X(A)$.
b) If $\tau_x \subseteq p$ then $p \in x^*$.
c) Let $U \subseteq X$, then U is a neighborhood of x if and only if $U \in p$ for each $p \in x^*$.
d) Let $A \subseteq X$. Then $x \in cl_X A$ if and only if $cl_{\beta X_d} A \cap x^* \neq \emptyset$.
e) For each $x \in X$, $\tau_x = \bigcap x^*$ is a filter and $cl_{\beta X_d}(x^*) = \overline{\tau}_x = \bigcap_{U\in\tau_x} cl_{\beta X_d} U$.
f) Let $A \subseteq X$, then x is an interior point of A if and only if $x^* \subseteq cl_{\beta X_d} A$. In particular, A is open if and only if $x^* \subseteq cl_{\beta X_d} A$ for each $x \in A$.

Proof. $a)$ Let $p \in \infty^*$, so $p \notin x^*$ for each $x \in X$. Hence $x \notin \bigcap_{A\in p} cl_X A$. Thus $x \notin cl_X A$ for some $A \in p$.

Conversely, suppose for each $x \in X$, there exists $A \in p$ such that $x \notin cl_X A$. Hence $p \notin B(X)$ and thus $p \in \infty^*$.
$b)$ Let $U \in p$ for each $U \in \tau_x$, thus $U \cap A \neq \emptyset$ for each $U \in \tau_x$ and for each $A \in p$. This implies $x \in cl_X A$ for each $A \in p$. Therefore $p \in x^*$.

$c)$ Let $U \in \tau_x$ and $p \in x^*$. Since $U \cap A \neq \emptyset$ for each $A \in p$, so $U \in p$.

Conversely, let $U \in \bigcap_{p\in x^*} p$ and $x \notin int_X(U)$. Then $x \in cl_X(U^c)$, there exists $p \in \beta X_d$ such that $U^c \in p \subseteq R$. This is a contradiction.
$d)$ By Theorem 2.3, there exists $p \in \beta X_d$ such that $A \in p \in x^*$. This implies that $cl_{\beta X_d} A \cap x^* \neq \emptyset$. Now let $cl_{\beta X_d} A \cap x^* \neq \emptyset$, so there exists $p \in x^*$ such that $p \in cl_{\beta X_d} A$. Thus $A \in p \in x^*$ and so $x \in cl_X A$.

$e)$ and $f)$ are obvious.

For every net $S = \{x_\alpha\}_{\alpha\in I}$ in a topological space X, the family $\mathcal{F}(S)$, consisting of all sets $A \subseteq X$ with the property that there exists $\alpha_\circ \in I$ such that $x_\alpha \in A$ whenever $\alpha \geq \alpha_\circ$, is a filter in the space X,(see Theorem 1.6.12 in [3]). So $\{\{x_\alpha : \alpha > \alpha_\circ\} : \alpha_\circ \in I\}$ has the finite intersection property.

Theorem 3.2. *Let X be a topological space. Then:*
(a) Let $\{x_\alpha\}_{\alpha\in I}$ be a net in X. If $x_\alpha \to p$ in βX_d for some $p \in x^*$. Then $x_\alpha \to x$ in (X, τ).
(b) Let $\{x_\alpha\}_{\alpha\in I}$ be a net in X and let $x_\alpha \to x$ in X. Then there exists $p \in x^*$ that is a cluster point of $\{x_\alpha\}_{\alpha\in I}$ in βX_d.
(c) Let $A \subseteq X$ be closed. Then A is compact if and only if $cl_{\beta X_d} A \cap \infty^* = \emptyset$.

Proof : (a) Let U be an open neighborhood of x. Then $U \in p$,(Lemma 3.1), and so there exists $\beta \in I$ such that $x_\alpha \in U$ for each $\alpha > \beta$. This implies $x_\alpha \to x$ in (X, τ).
(b) $\{\{x_\gamma : \gamma > \beta\} : \beta \in I\}$ has the finite intersection property then there exists $p \in \beta X_d$ that $\{x_\gamma : \gamma > \beta\} \in p$ when $\beta \in I$. Since $x_\alpha \to x$ in X. Thus for each open neighborhood U of x, there exists $\beta \in I$ such that $x_\gamma \in U$ when $\gamma > \beta$ and so $U \in p$. By Lemma 3.1 ii), implies $p \in x^*$. It is obvious p is cluster point of $\{x_\alpha\}_{\alpha\in I}$ in βX_d.

$c)$ Let $A \subseteq X$ is compact and $cl_{\beta X_d} A \cap \infty^* \neq \emptyset$, so there exists a net $\{x_\alpha\}_{\alpha\in I}$ in A such that $x_\alpha \to p$ for some $p \in cl_{\beta X_d} A \cap \infty^*$. Since A is compact so there is a subnet $\{x_\beta\}$ such that $x_\beta \to x \in A$ and so by (b), there is $q \in x^*$ such that $x_\beta \to q$. This implies $p = q$, and hence by Lemma 3.1 we have a contradiction.

Conversely, Let $cl_{\beta X_d} A \cap \infty^* = \emptyset$ and $A \subseteq X$ is not compact. Hence there is a net $\{x_\alpha\}_{\alpha\in I}$ in A such that any subnet of $\{x_\alpha\}_{\alpha\in I}$ is divergent in A. Since βX_d is compact, so there is a subnet $\{x_\beta\}$ such that $x_\beta \to p \in cl_{\beta X_d} A$. Also $p \in \infty^*$, because if $p \notin \infty^*$ then $p \in y^*$ for some $y \in X$. So by (a), $x_\beta \to y$ in X. This implies that $y \in cl_X A$ and this is a contradiction. Thus $p \in cl_{\beta X_d}(A) \cap \infty^*$ and we have a contradiction.

Theorem 3.3 (Robinson's Compactness). *Let*

(X,τ) be a topological space. Then $A \subseteq X$ is compact if and only if for every $p \in cl_{\beta X_d} A$ there exists $x \in A$ such that $p \in x^*$.

Theorem 3.4. *Let (X,τ_X) and (Y,τ_Y) be topological spaces. Then the following statements are equivalence.*
a) $f : X \to Y$ is continuous.
b) For each $x \in X$, $f^\beta(x^*) \subseteq (f(x))^*$.
c) $f^\beta : B(X) \to B(Y)$ is well defined and continuous.

Proof : a) implies b). Let $f : X \to Y$ is continuous, and pick $p \in x^*$. We must show $f^\beta(p) \in (f(x))^*$. By Lemma 2.1, we have

$$f^\beta(p) = \{A \subseteq Y : f^{-1}(A) \in p\}.$$

Now let $f^{-1}(A) \in p$ for some $A \subseteq Y$, so $x \in cl_X f^{-1}(A)$. Hence there exists a net $\{x_\alpha\} \subseteq f^{-1}(A)$ such that $x_\alpha \to x$ and so $f(x_\alpha) \to f(x)$. This implies $f(x) \in cl_Y A$ and hence $f^\beta(p) \in (f(x))^*$.

b) implies a). Let $f^\beta(x^*) \subseteq (f(x))^*$ for each $x \in X$. Let there exists a net $\{x_\alpha\}_{\alpha\in I}$ such that $x_\alpha \to x$ for some $x \in X$ and $f(x_\alpha)$ is not convergent to $f(x)$. Since $x_\alpha \to x$ so for each open neighborhood $U \in \tau_X$ of x there exists $\beta_U \in I$ such that $x_\alpha \in U$ for each $\alpha > \beta_U$. Thus $\mathcal{A} = \{\{x_\alpha : \alpha > \beta_U\} : x \in U \in \tau_X\}$ has the finite intersection property. Therefore there exists an ultrafilter p contains $\mathcal{A}$. It is obvious $p \in x^*$,(because for each $A \in p$ and for each open neighborhood U of x, we have $A \cap U \neq \emptyset$ so $x \in cl_X A$.)

Since $f(x_\alpha)$ is not convergent to $f(x)$, so there exists a sub net $\{x_\beta\}$ such that $x_\beta \to x$ and for some open neighborhood U of $f(x)$, $f(x_\beta) \notin U$ for each β. Since U is an open neighborhood of $f(x)$ so for each $B \subseteq Y$, if $f(x) \in cl_Y B$ then $U \cap B \neq \emptyset$. This implies $(f(x))^* \subseteq cl_Y U$, in particular, $U \in f^\beta(p)$ and hence $f^{-1}(U) \in p$, (see Lemma 2.1 and 3.1). Now we have $\{x_\beta : \beta\} \cap f^{-1}(U) \neq \emptyset$, and this is a contradiction.
a) and b) implies c). It is obvious.
c) implies a). Let $U \in \tau_Y$ then $cl_{\beta Y_d}(\tau_Y) \subseteq cl_{\beta Y_d}(U)$. This implies $cl_{\beta X_d}(\tau_X) \subseteq (f^\beta)^{-1}(cl_{\beta Y_d}(U)) = cl_{\beta X_d}(f^{-1}(U))$. This implies $f^{-1}(U) \in \tau_X$.

Theorem 3.5 (Open Mapping)**.** *Let (X,τ_X) and (Y,τ_Y) be two topological spaces, let $f : X \to Y$ be a continuous function and x^* be a closed subset of βX_d for each $x \in X$. Then f is open if and only if $(f(x))^* \subseteq f^\beta(x^*)$ for each $x \in X$.*

Theorem 3.6. *Let (X,τ_X) and (Y,τ_Y) be two topological spaces, let $f : X \to Y$ be a continuous function, and let K be a compact subset of X. Then $f(K)$ is compact.*

References

[1] W. Comfort and S. Negrepontis, *The Theory of Ultrafilters,* Springer-Verlag, Berlin, 1974.

[2] L. Gilman and M. Jerison, *Ring of Continuous Functions,* Van Nostrand, Princeton, N. J., 1960.

[3] R. Engelking, *General Topology,* Berlin, Sigma series in pure mathematics; Vol. 6, 1989.

[4] N. Hindman and I. Leader, *The Semigroup of Ultrafilters Near 0,* Semigroup Forum**59**(1999),33-55.

[5] N. Hindman and D. Strauss, *Algebra in the Stone-Čech Compactification, Theory and Application,* Springer Series in Computational Mathematics, Walter de Gruyter, Berlin, 1998.

[6] A. Robinson, *Nonstandard Analysis,* North-Holland, Amsterdam. 1966.

[7] M. A. Tootkaboni and T. Vahed, *The semigroup of ultrafilters near an idempotent of a semitopological semigroup,* Topology and its Applications , **V. 159, Issue 16,** (2012), 3494-3503.

[8] Y. Zelenyuk, *Ultrafilters and Topologies on Groups,* Walter de Gruyter, Berlin, 2011.

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ساختار جبروار لی بر $TM \oplus E$ القایی از E

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چکیده: جبروارهای لی متعدی، یک ساختار کلاف جبر لی بر هسته‌ی نگاشت لنگرشان القا می‌کنند. در این نوشتار، قصد داریم بررسی کنیم که عکس این مطلب برای حالت خاص $TM \oplus E$ تحت چه شرایطی برقرار است؛ به این ترتیب که یک ساختار کلاف جبر لی بر E ، تحت چه شرایطی به یک ساختار جبروار لی بر $TM \oplus E$ گسترش خواهد یافت.

کلمات کلیدی: جبروار لی، کلاف جبر لی، کروشه لی

بخش یک

تعریف [۱] : کلاف برداری $A \xrightarrow{\pi} M$ را یک جبروار لی [۱] می نامیم هرگاه دارای خواص زیر باشد:

۱. فضای برش های $\Gamma(A)$ دارای ساختار جبرلِی باشد.

۲. یک همریختی قوی کلاف‌های برداری $\rho : A \to TM$ (که نگاشت لنگر [۲] نامیده می‌شود) موجود باشد به گونه ای که یک همریختی جبرهای لی (که با همان علامت ρ نمایش داده می شود) از $\Gamma(A)$ به $\chi(M)$ القا کند .

۳. برای هر تابع هموار f روی M و برش‌های $s_1, s_2 \in \Gamma(A)$ تساوی زیر برقرار باشد :

$$[s_1, fs_2] = f[s_1, s_2] + (\rho(s_1)f)s_2$$

تعریف [۲] : جبروار لی $(A, \pi, M, \rho, [,])$ را متعدی [۳] گویند هرگاه ρ پوشا باشد.
فرض کنید $(A, \pi, M, \rho, [,])$ یک جبروار لی متعدی باشد. قرار دهید

$$K := ker(\rho)$$

بنابر [۳] K یک کلاف برداری است.کروشه موجود روی ΓA یک ساختار کلاف جبر لی بر K القا می کند [۲] .
قصد داریم بررسی کنیم که عکس این مطلب برای

[۱] Lie Algebroid
[۲] Anchor Map
[۳] Transitive

حالت خاص $TM \oplus E$ تحت چه شرایطی برقرار است؛ به این ترتیب که یک ساختار کلاف جبر لی بر E ، تحت چه شرایطی به یک ساختار جبروار لی بر $TM \oplus E$ گسترش خواهد یافت.

بخش دو

فرض کنیم یک E کلافِ جبر لی باشد. کلاف $TM \oplus E$ و نگاشت

$$\rho : TM \oplus E \longrightarrow TM$$

$$X \oplus T \mapsto X$$

را در نظر بگیرید.

سوال این است که چگونه می توان از ترکیب کروشه طبیعی $\chi(M)$ و کروشه موجود روی ΓE ، یک کروشه‌ی سازگار با ρ بر $TM \oplus E$ تعریف کرد؟

به طور طبیعی، اولین پاسخ، حاصل جمع این دو کروشه است؛ بنابراین قرار می دهیم:

$$[X \oplus T, Y \oplus S] := [X, Y] \oplus [T, S]$$

به سادگی دیده می شود که R ـ خطی بودن، پادمتقارن بودن و اتحاد ژاکوبی به کروشه‌ی جدید به ارث می‌رسد. لذا شرط سازگاری را بررسی می‌کنیم:

$$[X \oplus T, f(Y \oplus S)] = f[X \oplus T, Y \oplus S] + (X.f)(Y \oplus S)$$

یعنی حاصل‌جمع دو کروشه، زمانی سازگار خواهد شد که برای هر T, S در ΓE و X در $\chi(M)$ و f در $C^\infty(M)$ داشته باشیم:

$$[T, fS] = f[T, S] + (X.f)S$$

اما چنین رابطه‌ای قطعا برای هیچ کروشه‌ای برقرار نخواهد بود؛ بنابراین حاصل‌جمع، ترکیب مناسبی نیست. برای رفعِ مشکل، تعریف کروشه بر $TM \oplus E$ را به سه حالت تقسیم می‌کنیم:

حالت اول: انتظار داریم

$$[X \oplus \circ, Y \oplus \circ] = [X, Y] \oplus \circ$$

برای این حالت، شرط سازگاری برقرار است.

حالت دوم: انتظار می‌رود که

$$[\circ \oplus T, \circ \oplus S] = \circ \oplus [T, S]$$

شود. در این مورد، بررسی شرطِ سازگاری نتیجه می‌دهد:

$$[T, fS] = f[T, S] \qquad (۱)$$

حالت سوم: از آنجا که $\circ \oplus T$ در هسته‌ی ρ قرار دارد، انتظار داریم حاصل $[X \oplus \circ, \circ \oplus T]$ نیز در هسته‌ی ρ باشد، لذا:

$$[X \oplus \circ, \circ \oplus T] = \circ \oplus L(X, T)$$

برای پاسخ به سوال اصلی، باید دید تابع L باید چه شرایطی داشته باشد. این کار را در ۴ مرحله انجام می‌دهیم؛ دو مرحله اول از سازگاری با ρ و مراحل سوم و چهارم از اتحاد ژاکوبی نتیجه می‌شود.

مرحله اول:

$$\circ \oplus fL(X, T) = [f(X \oplus \circ), \circ \oplus T] = \circ \oplus L(fX, T)$$

و این یعنی L باید نسبت به مولفه اول $C^\infty(M)$ ـ خطی باشدیعنی

$$L(fX, T) = fL(X, T) \qquad (۲)$$

مرحله دوم:

$$\begin{aligned} \circ \oplus L(X, fT) &= [X \oplus \circ, f(\circ \oplus T)] \\ &= \circ \oplus fL(X, T) + (X.f)(\circ \oplus T) \end{aligned}$$

و این یعنی:

$$L(X, fT) = fL(X, T) + (X.f)T \qquad (۳)$$

مرحله سوم:

$$\begin{aligned}\circ = \ & [[X\oplus\circ, X'\oplus\circ], \circ\oplus T] \\ & +[[X'\oplus\circ, \circ\oplus T], X\oplus\circ] \\ & +[[\circ\oplus T, X\oplus\circ], X'\oplus\circ]\end{aligned} \qquad (۴)$$

بنابراین

$$L([X,X'],T) = L(X, L(X',T)) - L(X', L(X,T))$$

مرحله چهارم:

$$\begin{aligned}\circ = \ & [[\circ\oplus T, \circ\oplus T'], X\oplus\circ] \\ & +[[\circ\oplus T', X\oplus\circ], \circ\oplus T] \\ & +[[X\oplus\circ, \circ\oplus T], \circ\oplus T']\end{aligned} \qquad (۵)$$

لذا

$$L(X[T,T']) = [T, L(X,T')] + [L(X,T),T']$$

بنابراین با در اختیار داشتن کلاف جبری E با ویژگی ۱ و تابع L که در شرایط ۲ ، ۳ ، ۴ و ۵ صدق کند،

$$\begin{aligned}[X\oplus T, Y\oplus S] = \ & [X,Y]\oplus\Big([T,S] \\ & +L(X,S) - L(Y,T)\Big)\end{aligned}$$

یک کروشه سازگار با ρ روی $\Gamma(TM\oplus E)$ تعریف می‌کند.

مثال۱:

فرض کنیم ∇ هموستاری تخت بر خمینه M باشد. در این صورت با در نظر گرفتن $L(X,T)=\nabla_X T$ کروشه

$$\begin{aligned}[X\oplus T, Y\oplus S] = \ & [X,Y]\oplus\Big([T,S] \\ & +\nabla_X S - \nabla_Y T\Big)\end{aligned}$$

یک ساختار جبروار لی بر $TM\oplus L(E)$ تعریف می‌کند.

مثال ۲:

کلاف جبرلی $L(TM)$ با کروشه لی

$$[T,S] = T\circ S - S\circ T$$

را در نظر بگیرید. قرار دهید

$$L(X,T) = L_X T$$

با توجه به آنچه ذکر شد

$$\begin{aligned}[X\oplus T, Y\oplus S] = \ & [X,Y]\oplus\Big([T,S] \\ & +L_X S - L_Y T\Big)\end{aligned}$$

یک ساختار جبروار لی بر $TM\oplus L(TM)$ تعریف می‌کند.

مراجع

[1] E.Martinez, *Lagrangian mechanics on Lie algebroids*, Mathematicae, 67 (2001), 295-320.

[2] M. Boucetta, *Riemannian geometry of Lie algebroids*, Journal of the Egyptian Mathematical Society. Vol 19, 2011, pp 57–70.

[3] W. Greub, *Connections, Curvature, and Cohomology*,Vol I, Academic Press, Ch 2, p 84.

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λ-symmetry method for coupled second order ODEs

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Abstract: we present a procedure to find a first integral and consequently an integrating factor for a class of second order ODEs.

Keywords: Symmetry, λ-symmetry, First integral, Integrating factor.

1 INTRODUCTION

We obtain some of the foundational results about λ-symmetry, first integral and integrating factor for vector ODEs of second order $\ddot{\hat{r}} = \hat{f}$ where $\hat{r} = u\hat{\imath} + w\hat{\jmath}$ and $\hat{f} = F_1(x,u,w,\dot{u},\dot{w})\hat{\imath} + F_2(x,u,w,\dot{u},\dot{w})\hat{\jmath}$ also u and w are arbitrary functions, $\dot{u}$ denote the derivative of u with respect to x and $\dot{w}$ denote the derivative of w with respect to x. Using a theorem, we obtain a procedure to find an integrating factor $\mu_1(x,u,w,\dot{u},\dot{w})$ of $\ddot{u} = F_1(x,u,w,\dot{u},\dot{w})$ and an integrating factor $\mu_2(x,u,w,\dot{u},\dot{w})$ of $\ddot{w} = F_1(x,u,w,\dot{u},\dot{w})$ and consequently a first integral $I(x,u,w,\dot{u},\dot{w})$ for vector ODEs of second order $\ddot{\hat{r}} = \hat{f}$.

2 Integrating factor for coupled 2-order ODEs

In this section, we present a procedure to find an integrating factor and consequently a first integral for a vector ODEs of second order of the form $\ddot{\hat{r}} = \hat{f}$. We consider a vector ODEs of second order

$$\ddot{\hat{r}} = \hat{f} \tag{1}$$

where $\hat{r} = u\hat{\imath} + w\hat{\jmath}$ and $\hat{f} = F_1(x,u,w,\dot{u},\dot{w})\hat{\imath} + F_2(x,u,w,\dot{u},\dot{w})\hat{\jmath}$ also u and w are arbitrary functions, $\dot{u}$ denote the derivative of u with respect to x and $\dot{w}$ denote the derivative of w with respect to x. Therefore we have coupled second order ODEs of the form

$$\ddot{u} = F_1(x,u,w,\dot{u},\dot{w}), \tag{2}$$
$$\ddot{w} = F_2(x,u,w,\dot{u},\dot{w}). \tag{3}$$

where F_1 and F_2 are analytic functions of their arguments.

We denote by $A = \partial_x + \dot{u}\partial_u + \dot{w}\partial_w + F_1\partial_{\dot{u}} + F_2\partial_{\dot{w}}$ the vector field associated with a vector ODEs of second order of (1). Function $I(x,u,w,\dot{u},\dot{w})$ is a first integral for (1), such that $A(I) = 0$. An integrating factor of (2), is any function $\mu_1(x,u,w,\dot{u},\dot{w})$ such that $\mu_1(x,u,w,\dot{u},\dot{w})(\ddot{u} - F_1(x,u,w,\dot{u},\dot{w})) = D_x I(x,u,w,\dot{u},\dot{w})$ and an integrating factor of (3), is any function $\mu_2(x,u,w,\dot{u},\dot{w})$ such

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that $\mu_2(x,u,w,\dot u,\dot w)(\ddot w - F_2(x,u,w,\dot u,\dot w)) = D_x I(x,u,w,\dot u,\dot w)$.

The vector field $v = \xi(x,u)\partial_x + \eta(x,u)\partial_u$ is a λ-symmetry of $u^{(n)} = F(x,u^{(n-1)})$ if and only if

$$[v^{[\lambda,(n-1)]}, A] = \lambda.v^{[\lambda,(n-1)]} + \tau.A, \qquad (4)$$

for some $\lambda \in C^\infty(M^{(1)})$, $\tau = -(A+\lambda)(\xi(x,u))$ and $v^{[\lambda,(n-1)]} = \sum_{i=0}^{n-1}(D_x+\lambda)^i(1)\partial_{u_i}$ [1].

Using (4), the vector field $v_1 = \partial_u$ is a λ_1-symmetry of (2) if and only if

$$(A(\lambda_1)+\lambda_1^2-F_{1u}-\lambda_1F_{1\dot u})\partial_{\dot u}-(F_{2u}+\lambda_1F_{2\dot u})\partial_{\dot w} = 0, \qquad (5)$$

where $v_1^{[\lambda_1,(1)]} = \partial_u + \lambda_1\partial_{\dot u}$ and the vector field $v_2 = \partial_w$ is a λ_2-symmetry of (3) if and only if

$$(A(\lambda_2)+\lambda_2^2-F_{2w}-\lambda_2F_{2\dot w})\partial_{\dot w}-(F_{1w}+\lambda_2F_{1\dot w})\partial_{\dot u} = 0, \qquad (6)$$

where $v_2^{[\lambda_2,(1)]} = \partial_w + \lambda_2\partial_{\dot w}$.

Theorem 1. A system of the form

$$\begin{cases} I_x = \mu_1(\lambda_1\dot u - F_1) + \mu_2(\lambda_2\dot w - F_2), \\ I_u = -\mu_1\lambda_1, \\ I_w = -\mu_2\lambda_2, \\ I_{\dot u} = \mu_1, \\ I_{\dot w} = \mu_2, \end{cases} \qquad (7)$$

is compatibly for some functions $\lambda_1(x,u,w,\dot u,\dot w)$, $\lambda_2(x,u,w,\dot u,\dot w)$, $\mu_1(x,u,w,\dot u,\dot w)$ and $\mu_2(x,u,w,\dot u,\dot w)$, if and only if μ_1 is an integrating factor and $v_1 = \partial_u$ is a λ_1-symmetry of (2) also μ_2 is an integrating factor and $v_2 = \partial_w$ is a λ_2-symmetry of (3). In this case $I(x,u,w,\dot u,\dot w)$ is a first integral for a vector ODEs of second order of (1).

Proof: Let $I(x,u,w,\dot u,\dot w)$ be a first integral for a vector ODEs of second order of (1), then $\mu_1 = I_{\dot u}$ is an integrating factor of (2) and $\mu_2 = I_{\dot w}$ is an integrating factor of (3). Let $v_1 = \partial_u$ be a λ_1-symmetry of (2) then $A(I) = 0$ and $v^{[\lambda_1,(1)]}(I) = 0$ also, Let $v_2 = \partial_w$ be a λ_2-symmetry of (3) then $A(I) = 0$ and $v^{[\lambda_2,(1)]}(I) = 0$, hence we get

$$A(I) = 0 \Rightarrow I_x = -\dot u I_u - \dot w I_w - F_1\mu_1 - F_2\mu_2,$$
$$v^{[\lambda_1,(2)]}(I) = 0 \Rightarrow I_u = -\lambda_1 I_{\dot u} = -\lambda_1\mu_1$$
$$v^{[\lambda_2,(2)]}(I) = 0 \Rightarrow I_w = -\lambda_2 I_{\dot w} = -\lambda_2\mu_2.$$

Therefore we have the system of the form (7). We prove that, when (7) is compatible, necessarily $v_1 = \partial_u$ is a λ_1-symmetry and necessarily $v_2 = \partial_w$ is a λ_2-symmetry. The compatibility conditions between the equations (7), provide the following conditions

$$A(\lambda_1) = F_{1u} + \lambda_1F_{1\dot u} - \lambda_1^2 + \frac{\mu_2}{\mu_1}(F_{2u} + \lambda_1F_{2\dot u}) \qquad (8)$$

$$A(\lambda_2) = F_{2w} + \lambda_2F_{2\dot w} - \lambda_2^2 + \frac{\mu_1}{\mu_2}(F_{1w} + \lambda_2F_{1\dot w}) \qquad (9)$$

$$A(\mu_1) = -\mu_1(F_{1\dot u} - \lambda_1) + \mu_2F_{2\dot u}, \qquad (10)$$
$$A(\mu_2) = -\mu_2(F_{2\dot w} - \lambda_2) + \mu_1F_{1\dot w}, \qquad (11)$$
$$\lambda_1\mu_{1w} = -\lambda_{1w}\mu_1 + \lambda_2\mu_{2u} + \lambda_{2u}\mu_2, \qquad (12)$$
$$\mu_{1u} = -\lambda_{1\dot u}\mu_1 - \lambda_1\mu_{1\dot u}, \qquad (13)$$
$$\mu_{2w} = -\lambda_{2\dot w}\mu_2 - \lambda_2\mu_{2\dot w}, \qquad (14)$$
$$\mu_{1w} = -\lambda_{2\dot u}\mu_2 - \lambda_2\mu_{2\dot u}, \qquad (15)$$
$$\mu_{2u} = -\lambda_{1\dot w}\mu_1 - \lambda_1\mu_{1\dot w}, \qquad (16)$$
$$\mu_{1\dot w} = \mu_{2\dot u}. \qquad (17)$$

Using (8), we get $\Big(A(\lambda_1) + \lambda_1^2 - F_{1u} - \lambda_1F_{1\dot u}\Big)\mu_1 - \Big(F_{2u} + \lambda_1F_{2\dot u}\Big)\mu_2 = 0$, since $I_{\dot u} = \mu_1 \neq 0$ and $I_{\dot w} = \mu_2 \neq 0$, hence $\Big(A(\lambda_1)+\lambda_1^2-F_{1u}-\lambda_1F_{1\dot u}\Big)I_{\dot u} - \Big(F_{2u}+\lambda_1F_{2\dot u}\Big)I_{\dot w} = 0$, or correspond $\Big(A(\lambda_1)+\lambda_1^2 - F_{1u} - \lambda_1F_{1\dot u}\Big)\partial_{\dot u} - \Big(F_{2u} + \lambda_1F_{2\dot u}\Big)\partial_{\dot w} = 0$ and by comparing of (5), implies that $v_1 = \partial_u$ is a λ_1-symmetry of (2) also using (9) and by comparing of (6), implies that $v_2 = \partial_w$ is a λ_2-symmetry of (3).

Using (10) and (11), we get

$$\lambda_1 = \frac{A(\mu_1)}{\mu_1} - \frac{\mu_2}{\mu_1}F_{2\dot u} + F_{1\dot u}, \qquad (18)$$
$$\lambda_2 = \frac{A(\mu_2)}{\mu_2} - \frac{\mu_1}{\mu_2}F_{1\dot w} + F_{2\dot w}. \qquad (19)$$

In summary, a procedure to find integrating factors μ_1, μ_2 and consequently a first integral $I(x,u,w,\dot u,\dot w)$ for a vector ODEs of second order of (1) is as follows:

- Using (18) and (19), substituting λ_1 and λ_2 into (8),(9),(12)-(17) and solving them, we get, integrating factors μ_1 of (2) and μ_2 of (3).
- Substituting μ_1 and μ_2 into (18) and (19), we obtain, the functions λ_1 and λ_2, therefore the vector field $v_1 = \partial_u$ is a λ_1-symmetry of (2) and the vector field $v_2 = \partial_w$ is a λ_2-symmetry of (3).
- Substituting the functions μ_1, μ_2, λ_1 and λ_2 into the system (7) and solving them, we find a first integral $I(x, u, w, \dot{u}, \dot{w})$, for a vector ODEs of second order of (1).

Example 2.
We consider the Kepler problem in the $u-w$ plane, that is,

$$\ddot{\hat{r}} + \frac{\hat{r}}{r^3} = 0, \tag{20}$$

where $\hat{r} = u\hat{i} + w\hat{j}$ and $r = |\hat{r}|$. The respective equations of motions are

$$\ddot{u} = -\frac{u}{(u^2+w^2)^{\frac{3}{2}}} = F_1(x, u, w, \dot{u}, \dot{w}), \tag{21}$$

$$\ddot{w} = -\frac{w}{(u^2+w^2)^{\frac{3}{2}}} = F_2(x, u, w, \dot{u}, \dot{w}), \tag{22}$$

Using (18) and (19), substituting $\lambda_1 = \frac{A(\mu_1)}{\mu_1}$ and $\lambda_2 = \frac{A(\mu_2)}{\mu_2}$ into (8),(9),(12)-(17) and solving them, we get, $\mu_1 = w$ and $\mu_2 = u$ that are particular solutions of this equations. Substituting $\mu_1 = w$ and $\mu_2 = u$ into (18) and (19), we obtain, the functions $\lambda_1 = \frac{A(\mu_1)}{\mu_1} = \frac{\dot{w}}{w}$ and $\lambda_2 = \frac{A(\mu_2)}{\mu_2} = \frac{\dot{u}}{u}$, therefore the vector field $v_1 = \partial_u$ is a λ_1-symmetry of (21) and the vector field $v_2 = \partial_w$ is a λ_2-symmetry of (22). Substituting the functions $\mu_1 = w$, $\mu_2 = u$, $\lambda_1 = \frac{\dot{w}}{w}$ and $\lambda_2 = \frac{\dot{u}}{u}$ into the system (7) and solving them, we find a first integral $I(x, u, w, \dot{u}, \dot{w}) = w\dot{u} - u\dot{w}$, for a class of second order ODEs of (20).

References

[1] C. Muriel and J. L. Romero, "New methods of reduction for ordinary differential equation," IMA J. Appl. Math. vol. 66, pp. 111–125, 2001.

[2] C. Muriel and J. L. Romero, "C^∞-symmetries and reduction of equation without Lie point symmetries," J. Lie Theory, vol. 13, pp. 167–188, 2003.

[3] C. Muriel and J. L. Romero, "λ-symmetries and integrating factors," J. Non-linear Math. Phys, vol. 15, pp. 300–309, 2008.

[4] C. Muriel and J. L. Romero, "First integrals, integrating factors and λ-symmetries of second-order differential equations," J. Phys. A:Math. Theor, vol. 43, 2009.

[5] C. Muriel and J. L. Romero," Second-order ordinary differential equations and first integral of the form $A(t, x)\dot{x} + B(t, x)$," J. Nonlinear Math. Phys, vol. 16, pp. 209–222, 2009.

[6] V. Chandrasekar, M. Senthilvelan and M. Lakshmanan, Extended PrelleSinger method and integrability/solvability of a class of nonlinear nth-order ordinary differential equations, J. Nonl. Math.Phys. 12, pp. 184–201, 2005.

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Relation Between Cohomology of Topological Local Groups and Reduced Cohomology Topological Group

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Abstract: Let X be a T_2 topological local group, C Abelian topological group and topological group H is a enlargement of X. We prove that cohomology topological local group X is isomorphic with cohomology of quotient topological group H.

Keywords: Topological local group, Homomorphism of local groups,Topological sublocal group, Enlargeable, Monodrome, Contractive near-automorphism.

1 INTRODUCTION

Hu introduced reduced cohomology group on groups in the paper [2] and said relationships between it and cohomology groups and cohomology local groups. If every local p-map $f : V^p \to C$ of X into C for every $p > 0$ extends to a local p-map $f^* : X^p \to C$ defined throughout X^p and X act on C simply, then $H^p_*(X, C)$ is isomorphism to $H^p_L(X, C)$.

On the other word, Goldbring is said if X be a Hausdorff topological local group and φ open continuous map. and X extend to topological group H and let $\widetilde{\varphi} : H \to H$ be the unique extension of φ to and endomorphism of H then is a contractive near-automorphism and quotient H on union of $\ker\widetilde{\varphi}^n$ is locally isomorphic with topological local group X. **Now, we prove that cohomology topological local group is isomorphic with cohomology extended topological group of topological local group.**

For working we will use of theorem Goldbring [3]

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2 Primary Definitions

Definition 2.1. *Let X be a set with topology τ_X is called topological local group that satisfying the following condition:*

(i) *There is supposed to be an element $e \in X$ such that $e * x$ and $x * e$ exist for every $x \in X$ and $x * e = e * x$;*

(ii) $(\forall x, y \in X, V \in \tau_X)$ $xy \in V \Rightarrow (\exists U \in \tau_X)$ $y \in U$ *and* $xU \subseteq V$;

(iii) *The set* $D =: \{(x, y) \in X \times X | xy \in X\}$ *is a neighborhood of* (e, e) *in* $X \times X$, *and multiplication* $(x, y) \mapsto xy : D \to X$ *is continuous at* (e, e);

(iv) $X^{-1} = X$;

(v) *Inversion* $x \mapsto x^{-1} : X \to X$ *is continuous at* e.

(vi) $(\forall y \in X, V \in \tau_X)$ $y \in V \Rightarrow (\exists U \in \tau_X)$ $e \in U$ *and* $Uy \subseteq V$.

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Definition 2.2. *We say that a topological local group X is enlargeable if there exists a topological group G and a morphism $\phi : X \to G$ such that $\phi : X \to \phi(X)$ is a homeomorphism related to equivalent.*

Definition 2.3. *Let X and G are topological local groups, X operates on the left G means:*

There is a neighborhood X_1 of 1 in X and a neighborhood G_1 of 1 in G such that for every $x \in X_1$, $g \in G_1$ there exists $xg \in G$ with the following condition:

1. *xg is continuous on x and g;*
2. *We always have $1g = g$ For each $g \in G_1$.*
3. *If $g_1, g_2 \in G_1$ and $g_1 g_2$ is defined in G_1 then $xg_1 \in G_1$ is defined for all $x \in X_1$ $x(g_1 g_2) = (xg_1)g_2$*
4. *If $x_1, x_2 \in X_1$, $g \in G_1$ are so that $x_2 g$ and $x_1 x_2$ are defined on G_1 and X_1 respectively, $x_1(x_2 g) = (x_1 x_2)g$*

3 Cohomology of Topological local groups

In this section, we describe the cohomology of topological local groups $H^p(X, C)$. We shall use the non-homogeneous cochain exclusively. X is a topological local group, N a group and C is a center of N.
We are going to define a coboundry operator for every $p > 0$:

$$\delta : C_L^p(X, C) \to C_L^{p+1}(X, C)$$

Let $k \in C_L^p(X, C)$, then there is $f : V_0^p \to C$ such that $k = [f]$ such that V_0 be a neighborhood of 1 in X. such that for every $x \in V_0$, xc is define in neighborhood U_1 of $0 \in C$ and choose $V_1 \subset V_0$ and $V_1^p \subset f^{-1}(U_1)$. Let V_2 be a neighborhood $1 \in X$ such that $xy \in V_1$ for all $x, y \in V_2$. Now we define a locally $(p+1)$- map

$$\delta f(x_1, ..., x_2) = x_1 f(x_1, ..., x_{p+1}) + \sum_{i=0}^{p} (-1)^i f(x_1, ..., x_i x_{i+1}, ..., x_{p+1}) + (-1)^i f(x_1, ..., x_p)$$

For each $(x_1, ..., x_{p+1}) \in V_2^{p+1}$. We can be verified that the locally $(p+1)$- cochain $[\delta f]$ depends only on the given locally p- cochain k such that $\delta k = [\delta f]$ and δ is a homomorphism and $\delta\delta k = 0$.

3.1 Main Theorem

Lemma 3.1. *[3] Suppose X be a Hausdorff topological local group and φ open continuous map. Let H be enlargement of X and let $\widetilde{\varphi} : H \to H$ be the unique extension of φ to and endomorphism of H. Then the map $\widetilde{\varphi}$ is open. and for $F := \bigcup_n ker(\widetilde{\varphi}^n)$ we have :*

1. *F is a discrete normal subgroup of H and $\widetilde{\varphi}^{-1}(F) = F$;*
2. *$\widetilde{\varphi}$ descends to a contractive near-automorphism*

   $$\varphi_F : H/F \to H/F, \quad \varphi_F(xF) := \widetilde{\varphi}(x)F;$$

3. *for any symmetric open neighborhood $U \subseteq X$ of 1 with $U \times U \subseteq D$, the image $\pi(U)$ of U in H/F id open, and map*

   $$x \mapsto xF \quad : \quad U \to H/F$$

   is an isomorphism $X|_U \to (H/F)|_{\pi(U)}$ of topological local groups.

Hu introduced reduced cohomology group on groups in the paper [2] and said relationships between it and cohomology groups and cohomology local groups. If every local p-map $f : V^p \to C$ of X into C for every $p > 0$ extends to a local p-map $f^* : X^p \to C$ defined throughout X^p and X act on C simply, then $H_*^p(X, C)$ is isomorphism to

$H^p_L(X, C)$.

We want to use this subject and explain the follow theorem.

theorem 3.2. *We consider presupposition Lemma 3.1. Then $H^p_L(X, C)$ cohomology of local group X on abelian group C is isomorphism to cohomology of topological group $H^p_*(H/F, C)$ for all $p > 1$.*

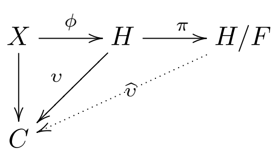
Proof. By Lemma 3.1 is an isomorphism

$$\pi \circ \varphi : X|_U \to (H/F)|_{\pi(U)}$$

of topological local groups then continuous open map $\pi \circ \varphi$ induces an isomorphism $(\pi \circ \varphi)^*$ on cohomology of topological local groups

$$H^p_L(X, C) \to H^p_L(H/F, C).$$

We have X is enlargeable to topological group H then for every topological group C and continuous map $X \to C$, by [4, Theorem 4.1] there exists unique continuous map $H \to C$ which commutes this diagram. F is a closed normal subgroup of H then H/F is a topological group and $\widehat{\upsilon}$ is a con-



tinuous map. Therefore local p-map $\pi(U) \to C$ extends to $H/F \to C$ then by [2, Theorem 12.4], we have reduce cohomology $H^p_*(H/F, C)$ is isomorphism wit local cohomology $H^p_L(H/F, C)$. So

$$H^p_L(X, C) \cong H^p_*(H/F, C).$$

□

References

[1] I.Goldbring, "Hilbert's fifth problem for local groups," Annales. Math, Vol. 172, No.2, pp 1269-1314, 2010.

[2] S.T. Hu, "Cohomology theory in Topological Groups," Michigan Math. J. **1** (1952), 11–59.

[3] D. Lou van den, D., I Goldbring, (1992, Aug.). "Locally compact contractine local groups". journal of Lie theory.

[4] H. Sahleh, A. Hosseini, "A Note on local groups", Journal of Mathematical Sciences: Advances and Applications, Vol. 14, No .2, pp 101-127, 2012

Books:

[5] J. R. Munkres, Topology, Second Edition. Asko K. Ghosh. Prentice-Hall, 2000.

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Functions with compact support in functionally countable subalgebra

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Abstract: Let $C_c(X)$ be the subalgebra of all real-valued continuous functions with countable image. We introduce and study $C_{ck}(X)$ as the set of all functions in $C_c(X)$ with compact support. We also define X_{cl}, as the subspace of X such that $X_{cl} = \cup_{f\in C_{ck}(X)}$suppf. It is shown that $C_{ck}(X)$ is isomorphic to $C_{ck}(Y)$, as pure objects if and only if X_{cl} homeomorphic with Y_{cl}.
Keywords: Zero-dimensional, Pure, Support.

1 INTRODUCTION

In this paper, all of spaces are zero-dimensional and hausdroff . $C(X)$ denote the ring of all continuous real-valued functions defined on X and $C_c(X)$ is subalgebra of all functions with countable range in$C(X)$. This ring i.e., $C_C(X)$ introduced in [4]. This paper is devoted to an investigation of subspace of X i.e., X_{cl} for which purity $C_{ck}(X)$ characterized through this subspace. At first, we state some preliminary definitions, results and symbols. Every ideal of $C_c(X)$ is denoted by I_C and $Z_C(f)$ is the zero set, for $f \in C_c(X)$, see [4]. $C_{ck}(X)$ and O^k_c are defined as follows: $C_{ck}(X) = \{f \in C_c(X) : CL_X coz f = supp f$ is compact$\}$

$$O^k_c = \{f \in C_c(X) : K \subseteq int_{\beta X} CL_X Z_C(f)\}$$

Ideal I_c of $C_c(X)$ is Pure if for each $f \in I_c$, there exists $g \in I_c$ such that $f = fg$ and in this case $g(x) = 1$ on support(f).
Maximal ideals of $C_{ck}(X)$ are precisely the sets $M^x \cap C_{ck}(X)$ for each $x \in X$, such that M^x is maximal ideal in $C_c(X)$, see ([3] and [9]).
Bkoueh in [2] proved that if X is locally compact, then $C_k(X) = \{f \in C(X) : supp f$ is compact$\}$ is pure. Abu-osba and Ezeh in [1] without assuming local compactness characterized purity of the ideal $C_k(X) \subseteq C(X)$ using the subspuce X_l i.e., the set of all points in X with compact neighborhood. These interesting results motivates us to introduce and study $C_{ck}(X)$ and C_{cl} in this the present paper. We denote $C_{ck}(X) = \bigcup_{f\in C_{ck}(X)} supp f$, so $X_{cl} \subseteq X_l$, if $C_K(X)$ is pure (see[1]).

2 The properties of X_{cl} and $C_{ck}(X)$

The following results investigate same properties $C_{ck}(X)$ and relations between X_{cl} and $C_{ck}(X)$.

Lemma 2.1. *$C_k(X)$ is an ideal in $C(X)$.*

Proof. Let $f, g \in C_k(X)$, $z(f + g) \supseteq z(f) \cap z(g)$, hence $CL_X coz(f + g) \subseteq CL_X(X - (z(f) \cap z(g)) =$

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$CL_X(X - z(f)) \cup (X - z(g)) = CL_X(cozf \cup cozg) = CL_X cozf \cup CL_X cozg$ is compact. So $CL_X coz(f+g)$ is compact, since X is Hausdroff and $f, g \in c_k(X)$,SO $f + g \in C_k(X)$. We show that $f.g \in C_k(X)$, for every $f \in C(X)$ and $g \in C_k(X)$. For this, $CL_X(X - z(f.g)) = CL_X(X - z(f) \cup z(g) = CL_X(cozf) \cap CL_X cozg \subseteq CL_X cozg$. By suppose $CL_{(X)} cozg$ is compact. Since $CL_x(cozf) \cap CL_x(cozg)$ is closed and X is Hausdroff, so $CL_X cozf.g$ is compact. Hence $f.g \in C_K(x)$. □

Theorem 2.2. *If I is an ideal in $C(X)$, then $I_c = I \cap C_c(X)$ is an ideal in $C_c(X)$.*

Proof. Let $f.g \in I_c$, since I is a ideal of $C(X)$ $f + g \in I$. But $Im(f + g) \subseteq Imf + Img$. by suppose $|Imf + Img| \leq \aleph_\circ, s \circ |Im(f + g)| \leq \aleph_\circ$ i.e., $f + g \in C_c(X)$. Consequently $f + g \in I_c$. But, if $f \in I_C$, $g \in C_c(X)$, then $f.g \in C_c(X)$ and since $f \in I$ and I is a ideal $C(X)$ we infer that $f.g \in I$ and we are done. □

Corollary 2.3. *$C_{ck}(X)$ is an ideal of $C_c(X)$.*

Proof. By Theorem 2.2 and this fact that $C_{ck}(X) = C_k(X) \cap C_c(X)$. □

Definition 2.4. *Ideal I_c in $C_c(X)$ is called a Z_c-ideal if $Z(f) \in Z(I_c)$ then $f \in I_c$, for every $f \in C_c(X)$ or the otherhand $Z^{-1}(Z(I_C)) = I_C R$.*

Lemma 2.5. *$C_{CK}(X)$ is a Z_C ideal.*

Proof. Let, $Z(h) = Z(f) for every f \in C_{ck}(X)$ and $h \in C_c(X)$. Then, $suppf = CL_x(X - Z(f)) = CL_x(X - Z(g) = suppg$ is compact so $g \in C_{ck}(X)$. r

□

Lemma 2.6. *Let φ be a ring and I an ideal in φ. Then an ideal A of I is prime in I if and only if $A = I \cap P$, for some P, prime in φ. furthermore, p is unique if A is proper.*

Proof. See[7]. □

Corollary 2.7. *Let I be an ideal of $C(X)$, then its maximal ideals are precisely the ideals of the form $I \cap M$, where M is maximal in $C(X)$ and M does not contain.*

Proof. Let, $A = M \cap I, M \not\supset I$ and M is maximal ideal in $C(X)$. Then $\frac{I}{A} = \frac{I}{M \cap I} \cong \frac{M+K}{M} = \frac{C(X)}{M}$ is field, which implies A is a maximal ideal of I. Conversely, Suppose A is a Max ideal of I. By Lemma 2.6. there exist M prime in $C(X)$ such that $A = I \cap M$. We claim that M is Maximal ideal in $C(X)$. To see this, assume there exist an ideal M' in $C(X)$ with $M \subsetneq M' \subsetneq C(X)$. Let $s \in C(X) \setminus M'$, the residue class ring $\frac{I}{A}$ is field, with unity $e + A$ say, then $e^2 s + A = es + A$, which implies $e(e - s) \in A$. Since M is prime and $e \notin M$, so $es - s \in M$. Since $A = I \cap M \subset I \cap M' \subset I$ and A is Maximal in I we infer that $I \cap M' = A$ or $I \cap M' = I$. But $es - s \in M'$, hence $es \notin M'$, so we must that $I \cap M' = A$. Now let $x \in M'$ then $xe \in I \cap M' = A$, which implies that $xe \in M$. Hence $x \in M$. This shows $M' = M$.

The uniqueness follows from lemma 2.6. Previous results, we share for $C_c(X)$.

Lemma 2.8. *let I_c is an ideal in $C_c(X)$, then. its maximals are precisely the ideals of the form $I_c \cap M_c$, whence M_c is maximal in $C_c(X)$ and M_c does not contain.*

Proof. Since is a commutative ring with unity. see[3] □

Lemma 2.9. *$X_{cl} = \phi$ if and only if $C_{ck}(X) = (0)$.*

Proof. let $X_{cl} = \phi$ i.e., $\bigcup_{f \in C_{ck}} suppf = \phi$ .so $suppf = \phi$ for every $f \in C_{ck}(X)$. Hence $cozf = \phi$ for every $f \in C_{ck}(X)$ i.e., $f = 0$ so $C_{CK}(X) = (0)$. Conversely, It is evident, $f = 0$, then $suppf = \phi$. Hence $X_{cl} = \phi$. □

Remark 2.10. *We introduced subspace $X_L \subseteq X$ in section 1, the set of all points in X with compact neighborhood (nhood). But, in the general case, $X_l \neq X_{cl}$. Here we give an example of a space*

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such that $X_{cl} \neq X_L$. In [1],we has be shown that $X_L = coz(C_k(X))$ so X_L is an open subspace of X,

Example 2.11. *Let $X = [-1, 1]$ with all points are isolated except for $x = 0$ has its usual nhoods. Then $X_L = X - \{0\}$. It is enough ,let be*

$$f(x) = \begin{cases} 1 & x = a \\ 0 & otherwise \end{cases}$$

for every $a \in X - \{\circ\}$, $f \in C_k(X)$. Now , let

$$g(x) = \begin{cases} x & x = \frac{1}{n}, n \in Z^* \\ 0 & otherwise \end{cases}$$

then $suppg = \{\frac{1}{n}, n \in Z^*\} \cup \{\circ\}, g \in C_{ck}(X)$ i.e $0 in X_{cl}$. Hence $X_l \neq X_{cl}$.

3 Relation between purity $C_{ck}(X)$ and X_{cl}

Brouche in [3] proved that if X is locally compcat, then $C_k(X)$ is a pure ideal. Abu osba and AL. Ezeh in [1] have shown that there are non-locally compact spaces X such that $C_k(X)$ is a non-trivial pure ideal. in this paper,According to this we study purity of $C_k(X)$ and it's Relationship with X_{cl}. At first, we give an example of a pure ideal.

Example 3.1. *Let $X = Q$ with all points have their usual nhoods except for $x = \circ$ is isolated then $C_{ck}(X) = \{f \in C_c(X) : f = 0$ except for $x = 0\}$ Since Q is countable, so $C_C(X) = C(X)$. Since if there exist $f \in C_k(X)$ such that $f(x) \neq f(0)$ and $x \in cozf$, there are a sequence of rational numbers, $\{r_n\}$ say, that $\{r_n\}$ is convergence to x, in this case, suppf is not compact i.e $f \notin C_{ck}(X)$. We show that $C_{CK}(X)$ is pure. Let $g(0) = 1$ and otherwise $g = 0$.Then $g \in C_{CK}(X)$and $f.g = f$,for every $f \in C_{CK}(X)$.*

Theorem 3.2. *Let I_c be a Z_c ideal contained in $C_{ck}(X)$ then I_c is pure ideal if and only if $cozI_c = \cup_{f\in I_c} suppf$.*

Proof. Let I_c be pure ideal it is clear that $cozI_c = \cup_{f\in I_c} cozf \subseteq \cup_{f\in I_c} suppf$. Now, if $f \in I_c$, then there exists $g \in I_c$ such that $f = fg$. so $g = 1$ on suppf. Thus $suppf \subseteq cozg$. Hence $cozI = \cup_{f\in I_c} suppf$.
$Cnversely, let$coz $I_c = \cup_{f\in I_c} suppf$. if $g \in I_c$, then $suppg \subseteq \cup_{f\in I_c} suppf = cozI_c = \cup_{f\in I_c} cozf$. So $suppg \subseteq \bigcup_{i=1}^{n} cozf_i$, for $f_1, f_2, \ldots, f_n \in I_c$ since $I \subseteq C_{ck}(X)$ and suppg is compact. Let $h = \sum_{i=1}^{n} f_i^2$. Then $h \in I_c$ and $cozh = \bigcup_{i=1}^{n} cozf_i$. Whence X is zero dimensional and suppg is compact, we have $k(suppg) = 1$ and $k(Z(h)) = 0$, for a $k \in C_c(X)$. Hence $g = gk$ and $Z(h)) \subseteq Z(k)$. therefore $k \in I_c$, since I_c is a Z_c ideal .Thus is pure ideal of $C_{ck}(X)$.

Corollary 3.3. *X_{cl} is subspace open of X if $C_{ck}(X)$ is pure..*

Proof. Since $coz(C_{ck}(X)) = \bigcup_{f\in C_{ck}(X)} suppf = X_{cl}$. □

Theorem 3.4. *Let $C_{ck}(X)$ be pure ideal then for each proper ideal I_c of $C_{ck}(X)$, $cozI_c$ is contained properly in X_{cl}.*

Proof. Suppose $I_c \subset C_{ck}(X))$, Evidently $cozI_c \subseteq X_{ck}$ .We show that $cozI_c \neq X_{cl}$. Otherwise $cozI_c = X_{cL}$. Let $f \in C_{ck}(X)$, Since $C_{ck}(X)$ is pure ideal we infer that $suppf \subseteq X_{cl} = cozI_c$. Hence $suppf \subseteq \bigcup_{i=1}^{n} f_i$,where $f_i \in I_c$ for each i. Suppose $g = \sum_{i=1}^{n} f_i^2$. Then $g \in I_c$, and $cozg = \bigcup_{i=1}^{n} cozf_i$ Define

$$h(x) = \begin{cases} \frac{f}{g}(x) & x \in cozg \\ 0 & \text{otherwise} \end{cases}$$

then $h \in C_c(X)$, since $suppf \subseteq cozg$. Moreover $f = gh \in I_c$. So $C_{ck}(X) = I_c$ □

Remark 3.5. *If $C_{ck}(X)$ is not pure, then theorem 3-9 need not be true. Please note the following example.*

Example 3.6. *Let $X = [-1, 1] \cup \{\frac{1}{n}, n \in N\}$ with the subspace topology. Then*

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$C_{ck}(X) = \{ f \in C_c(X) : cozf \subseteq \{\frac{1}{n},\ n \in N\}$. $C_{ck}(X)$ is not pure ideal of $C_c(X)$, since $\circ \notin cozC_{ck}(X)$ but $\circ \in X_{cl}$ .Because if $cozf = \{\frac{1}{n}, n \in N\}$, then $suppf = \{\frac{1}{n}, n \in N\} \cup \{\circ\}$

Remark 3.7. *whence the maximal ideals of $C_{ck}(X)$ are precisely the sets $M_c \cap C_{ck}(X)$ (see corollary 2.7) . So $M_C^x \cap C_{ck}(X)$ is fixed maximal ideal of $C_{ck}(X)$, for every $x \in X_{cl}$ and M_c^x, fixed maximal ideal in $C_c(X)$, such that $M_c^x \not\supseteq C_{ck}(X)$. But the latter relation is true, because if $M_C^x \supseteq C_{ck}(X)$, then $f(x) = 0$, for every $f \in C_{ck}(X)$. So $x \in \cap_{f \in C_{ck}(X)} Z(f)$ i.e., $x \notin cozc_{ck}(X) = X_{cl}$. This is a contradiction.*

Finally using the previous results, we get our the main result by the following theorem.

Theorem 3.8. *Let $C_{ck}(X)$ and $C_{ck}(Y)$, be pure ideals. Then X_{cl} is homeomorphic to Y_{cl} if and only if $C_{ck}(X)$ is isomorphic to $C_{ck}(Y)$.*

Proof. If $C_{ck}(X)$ is isomorphic to $C_{ck}(Y)$, then X_{cl} is homeomorphic to Y_{cl} . Whence $C_{ck}(X) \approx C_{ck}(Y)$ and $\{M_c^x \cap C_{ck}(X) : x \in X_{cL}\} \simeq X_{cl}$. (set of maximal ideals admits the hull-kernel topology). Similarly $\{M_c^x \cap C_{ck}(Y) : y \in Y_{cl}\} \simeq Y_{cl}$ □

Conversely, suppose $\varphi : X_{cl} \to Y_{cl}$ is a homeomorphism. Let $f \in C_{ck}(Y)$,then $f_1 \circ \varphi \in C_(X_{cL})$, where $f_1 = f\ |_{Y_{cl}}$. But $cozf = \varphi(cozf_1 \circ \varphi)$, since $cozf \subset Y_{cl}$. Which implies that $\varphi^{-1}(cozf) = cozf_1 \circ \varphi$. Therefore $CL_{X_{cl}} cozf_1 \circ \varphi = CL_{X_{cl}} \varphi^{-1}(cozf) = \varphi^{-1}(CL_y cozf) = \varphi^{-1}(suppf)$. Since suppf is contained in Y_{cl}, for each $f \in C_{ck}(X)$, define $g_f : X \to R$ by

$$g_f(x) = \begin{cases} f_\circ \varphi(x) & x \in X_{cl} \\ 0 & x \in X - \varphi^{-1}(suppf) \end{cases}$$

X_{cl} is open, (here we used purity of $C_{ck}(X)$) and since $suppg_f = CL_{x_{cl}} cozf_1 \circ \varphi$ is compact. Hence $g_f \in C_{ck}(X)$. Now, define $\bar{\varphi} : C_{ck}(Y) \to C_{ck}(X)$ by $\bar{\varphi}(f) = g_f$, then $\bar{\varphi}$ is a ring homomorphism.

Now, we show that $\bar{\varphi}$ is one to one and onto. Suppose $\bar{\varphi}(f) = 0$, then $(f_1 \circ \varphi)(x) = \circ$ for every $x \in X_{cl}$. So $cozf_1 \circ \varphi = \bar{\varphi}(cozf) = \phi$, i.e., $f = \circ$. Let $f \in C_{ck}(X)$. Define $g : Y \to R$ by

$$g(y) = \begin{cases} f \circ \varphi^{-1}(y) & y \in Y_{cl} \\ \circ & y \in Y - \varphi(suppf) \end{cases}$$

Then $f \in C_c(X)$. (We used here purity $C_{ck}(X)$).

Moreover if $g(y) \neq \circ$, then $\varphi^{-1}(y) \in cozf$. So cozg$\subseteq \varphi(cozf)$. As a consequence $CL_{Y_{cl}}(cozg) \subseteq CL_{Y_{cl}} \varphi(cozf) = \varphi(CL_X(cozf) = \varphi(suppf)$. But $CL_{Y_{cl}}(cozg) = CL_Y cozg = suppg$. Since suppg is compact, thus $g \in C_{ck}(Y)$. Now

$$\bar{\varphi}(g)(x) = \begin{cases} g_1 \circ \varphi(x) & x \in X_{cl} \\ \circ & x \in X - \varphi^{-1}(suppg) \end{cases}$$

$$= \begin{cases} f \circ \varphi^{-1} \circ \varphi(x) & x \in X_{cl} \\ \circ & x \in X - \varphi^{-1}(suppg) \end{cases}$$

=f(x)
so $\bar{\varphi}(g) = f$, i.e $\bar{\varphi}$ is onto. So $C_{ck}(X)$ is ring isomorphic to $C_{ck}(Y)$.

Corollary 3.9. *If $C_{ck}(X)$ is pure ideal, then $C_{ck}(X)$ is isomorphic to $C_{ck}(X_{cl})$.*

Proof. It is sufficient t, let $Y = X_{cl}$ in Theorem 3.8. □

4 Result

In this article, we prove that for every zero-dimensional space X, topological properties subspace X_{cl} of X characterize in terms of a suitable algebraic property of $C_{ck}(X)$ as an ideal of $C_c(X)$. We conclude X_{CL} is a subspace of X with locally compact neighborhood. This neighborhood is created by suppf, for some $f \in C_c(X)$.

References

[1] E. A. Abu Osba and H. AL. Ezeh, "Purity of the ideal of continuous functions with compact support," Math. J. Okayama Univ. (41) (1999), 111-120.

[2] R. Bkouche, Purete, Mollesse et paracompact, Acad. Sci. paris Ser. A.270. 1653-1655 (1970)

[3] J. G. Brookshear, " projective ideals in rings of continuous functions," Pacific. J, Math. 71 (2) (1977), 313-333 .

[4] M. Ghadermazi, O. A. S. Karamzadeh, and M. Namdari, "On functionally countable subalgebra of $C(X)$," Rocky Mountain J. Math. (29) (2011) 1153-1164.

[5] L. Gillman and M. Jerison, "Rings of continuous functions," Springer, 1967.

[6] C. Kohls, "Ideals in rings of continuous functions", Fund. Math. (45) (1957), 28-50

[7] N. H. Mc Coy, prime ideals in general rings, Amer Math, 45 (1957), 22-27.

[8] J. D. Mc Kinght,"On the characterization of ring of functions", Doctoral thesis, purdue university, Ind. 1953.

[9] D. Rudd,"On isomorphisms between ideals I rings of continuous functions", Trans. Amer. Math. Soc. (159) (1971), 335-353.

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On the preservation of Volterra spaces

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Didicated to memory of Prof. Zbigniew Piotrowski

Abstract: We will investigate the preservation of Volterra spaces under cartesian product. We also discuss about conditions which guarantees Volterraness under images and preimages of special mappings.

Keywords: Volterra spaces, weakly Volterra spaces, feebly open mappings, feebly continuous function.

1 INTRODUCTION

A topological space X is said to be a *Volterra space* (resp. *weakly Volterra*) [1], [2] if any finite intersection of dense G_δ subsets is dense (resp. nonempty). The name refers to a paper of Vito Volterra [3] who proved that if $f : \mathbb{R} \to \mathbb{R}$ is any function such that both $C(f) = \{x \in \mathbb{R} : f \text{ is continuous at x}\}$ and $D(f) = \mathbb{R} - C(f)$ are dense in $\mathbb{R}$, then there is no function $g : \mathbb{R} \to \mathbb{R}$ such that $C(g) = D(f)$ and $D(g) = C(f)$. Clearly, every Baire space is Volterra, but the converse is not true in general. In fact, if $X = [0, \infty)$ with topology having basis $\{[a, \infty) - F : a \in X \text{ and F is a finite subset of X}\}$, then X is not of second category, because it is the union of countably many closed, nowhere dense sets $\{[0, n) : n \in \mathbb{N}\}$. However, each dense G_δ-set is of the form $[a, \infty) - C$, where $a \in X$ and C is a countable subset. Therefore, the intersection of any pair of such sets is also is dense. Hence, X is Volterra. Gruenhage and Lutzer [3] have shown that a space that is T_3 and has a dense metric subspace or is T_3, metacompact, first countable and is a σ-space is Volterra if and only if it is Baire.

Let $X = \mathbb{R}^- \cup \mathbb{Q}^+$ with topology inherited from the reals, where $\mathbb{R}^-$ denoted the non-positive reals and $\mathbb{Q}^+$ the non-negative rationales, then X is the union of dense G_δ sets

$$A_1 = \mathbb{R}^- \cup \left\{ \frac{p}{q} : \ (p, q) = 1 \ \textit{and} \ q \ \textit{is even} \right\}$$

and

$$A_2 = \mathbb{R}^- \cup \left\{ \frac{p}{q} : \ (p, q) = 1 \ \textit{and} \ q \ \textit{is odd} \right\}.$$

But $A_1 \cap A_2 = \mathbb{R}^-$ is not dense in X. Thus X is not a Volterra space. Since X contains a Baire subset R^-, the intersection of any countable G_δ dense subset of X is nonempty. So that it is a weakly Volterra space. Hence, the class of weakly Volterra spaces is strictly larger than the class of Volterra spaces.

It is natural to ask under what condition(s) a function, it preserves Volterra (or weakly Volterra)

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property under image or preimage? Unfortunately, even continuous surjections do not preserve Volterra (and weakly Volterra) property under images. For example, the space of rational numbers (which is not even weakly Volterra space) is the continuous image of any countable set with discrete topology, which is a Baire space. In the next section, we will investigate stability of Volterra and weakly Volterra property under images and preimages of some mappings.

2 RESULTS

It is known that Baire Category is not preserved under cartesian product. In 1998, Gauld et al. [1] provided an example of a metric Baire space X such that $X \times X$ it is not even weakly Volterra. We recall that a topological space X is said to be *resolvable* if it contains two disjoint and dense subsets. Cao and Junnila [4] have show that the product space X×Y is not weakly Volterra if X is a Volterra space of the first Baire category and Y is dense-in-itself separable metric space. Indeed, product of a Volterra space which is of the first category and a resolvable space that the points are G_δ can not be a weakly Volterra space.

Theorem 2.1. *Suppose that X is a Volterra space which is not of the first category and Y is a resolvable space with G_δ-set points. Then the product space $X \times Y$ is not weakly Volterra space.*

Proof. There exists a decreasing sequence $\{G_n : n \in \mathbb{N}\}$ of dense open subsets of X such that $\bigcap_{n\in\mathbb{N}} G_n = \emptyset$. As Y is resolvable, there exist disjoint dense subsets A and B in Y. Fix $n \in \mathbb{N}$. The sets $A' = G_n \times A$ and $B' = G_n \times B$ are disjoint and dense in $X \times Y$. Moreover, the family $\{G_n \times Y : n \in \mathbb{N}\}$ is point-finite and open in $X \times Y$ and each set of the form $G_n \times \{y\}$ is a G_δ-set in $X \times Y$. Hence, as $A' = \bigcup_{a\in A} G_n \times \{a\}$ and $B' = \bigcup_{b\in B} G_n \times \{b\}$, it follows that A' and B' are G_δ-sets [4] which are disjoint and dense. □

However, if product space of $X \times Y$ is Volterra, then so is X and Y.

Theorem 2.2. *(i) If product space of $X \times Y$ is Volterra, then so is X and Y.*

(ii) If product space of $\{X_\alpha : \alpha \in A\}$ is Volterra, then so is X_α ($\forall \alpha \in A$).

Proof. (i) Let A and B be two dense G_δ subset of X. Then $A \times Y$ and $B \times Y$ are two dense G_δ set in $X \times Y$ and so $A \times Y \cap B \times Y = (A \cap B) \times Y$ is dense. Therefore $A \cap B$ dense in X.

(ii) Fix $\alpha \in A$. Let A and B be two dense G_δ subset of X_α. Then $A \times \prod_{\beta\neq\alpha} X_\beta$ and $B \times \prod_{\beta\neq\alpha} X_\beta$ are dense in $\prod_{\beta\in A} X_\beta$ and so is $(A \cap B) \times \prod_{\beta\neq\alpha} X_\beta$. Thus $A \cap B$ dense in X_α. (Note that $\overline{\prod_{\alpha\in A} X_\alpha} = \prod_{\alpha\in A} \overline{X_\alpha}$). □

A space X is called *P-space* if every G_δ-subset of X is open. The space X called *almost P-space* if each non-empty G_δ-subset of X has non-empty interior.

Note that, there is a hereditarily Baire space and a hereditarily Volterra space such that the product of them is not even weakly Volterra. Indeed, consider $\mathbb{R}^d$ the real numbers with the density topology and let $X = \{f \in 2^{\omega_1} : f^{-1}(1) < \omega\}$ with the topology inherited from the countably supported product topology on 2^{ω_1}. Then each subset of $\mathbb{R}^d$ is Baire and X is P-space. But $\mathbb{R}^d \times X$ is not weakly Volterra [5].

A function f from X to Y is called

(a) *feebly open* if for every open subset U of X, $int(f(U))$ is nonempty.

(b) *quasi open* if for every open set $V \subseteq X$, $f(U) \subseteq \overline{int f(U)}$.

(c) *feebly continuous* if for every open subset W of Y, $int(f^{-1}(W))$ is nonempty.

(d) *feeble homeomorphism* if it is feebly open and feebly continuous bijection.

(e) *quasi-continuous* if for each open set $V \subseteq Y$, we have $f^{-1}(V) \subseteq \overline{int f^{-1}(V)}$.

Frolik [6] proved that if f is both continuous and open mapping from a Baire space X onto a topological space Y, then Y is Baire. In 1977, Neubrunn improved Frolik's theorem by showing that the last result remains true if f only assumed to be quasi-continuous and feebly open.

A similar result for Volterra and weakly Volterra spaces has been proved in [1]:

Theorem 2.3. *([1, Proposition 5.1]) Suppose that $f : X \to Y$ is continuous and feebly open. If X is weakly Volterra then so is Y. If X is Volterra and f is surjective then Y is Volterra.*

In the case of Baire spaces, quasi-continuity of f is sufficient. The following example shows that we need the full strength of continuity and feeble openness [1]:

Example 2.4. *Let $X = \mathbb{N}$, O denote the odd positive integers and E be the even positive integers. For each pair (m, n) of positive integers let*

$$U_{m,n} = \{x \in X : eighter \quad x \geq 2m-1 \quad and \quad x \in O \quad or \quad x \geq 2n \quad and \quad x \in E\}.$$

Then $\{\emptyset\} \cup \{U_{m,n} : m, n \in \mathbb{N}\}$ defines a topology on X. It is readily checked that the two subsets O and E are G_δ and as subspaces are Volterra. From $O \cap E = \emptyset$, it follows that X is not weakly Volterra.

Define $f : E \to X$ by $f(e) = \frac{e}{2}$. The function f is open because if $U_{m,n} \cap E$ is open in E then $f(U_{m,n} \cap E) = U_{\frac{n}{2}+1,\frac{n}{2}}$ if $n \in E$ and $f(U_{m,n} \cap E) = U_{\frac{n+1}{2},\frac{n+1}{2}}$ if $n \in O$. Also, f is quasi-continuous because for any open $U_{m,n} \subset X$, let $k = max\{2n, 2m-1\}$. Then $U_{1,k} \cap E \subset f^{-1}(U_{m,n})$ and since $U_{1,k} \cap E$ is dense in E it follows that $int(f^{-1}(U_{m,n}))$ is dense in $f^{-1}(U_{m,n})$. However, E is Volterra but X is not even weakly Volterra. Therefore quasi-continuous open bijections do not preserve Volterra property.

The next result gives a sufficient condition for preserving Volterra and weakly Volterra property under preimages.

Theorem 2.5. *Let $f : X \to Y$ be a one-to-one, open and feebly-continuous function from topological space X to Volterra (resp. weakly Volterra) space Y. Then X is Volterra (resp. weakly Volterra) space.*

Proof. Let A and B be two dense G_δ-subset of X. Since f is one-to-one and open, $f(A)$ and $f(B)$ are G_δ-set in Y. Moreover, $f(A)$ and $f(B)$ are dense in Y, as if V be a non-empty open subset of Y, then by almost continuity of f $int(f^{-1}(V)) \cap A \neq \emptyset$ and so $f(A) \cap V \neq \emptyset$. Similarly, $f(B)$ is dense in Y.
As Y is Volterra space, $f(A) \cap f(B)$ is dense. Let $G \subset X$ be a non-void open set. Then $f(G)$ is open and intersects $f(A) \cap f(B) = f(A \cap B)$. Therefore $A \cap B \cap G \neq \emptyset$. So X is Volterra space. A similar argument shows that if Y be weakly Volterra, then so is X. The following example shows that Volterra (resp. weakly Volterra) property may not be preserved under preimage of continuous open (or closed) continuous functions. □

The following example [1] shows that bijectivity is necessary in Theorem 2.3.

Example 2.6. *Let $X = [0, 2] \cap \mathbb{Q}$ and $Y = \{0, 1\}$. Define $f : X \to Y$ by $f(x) = 0$ if $x \leq \sqrt{2}$ and $f(x) = 1$ if $x \geq \sqrt{2}$. Then f is continuous, open and closed but Y is Volterra whereas X is not weakly Volterra; however both spaces are metric.*

In our proof of Theorem 2.3, we need the full strength of openness:
If A be G_δ-set in X, then $f(A) = f(\bigcap_{n\in\mathbb{N}} A_n) = \bigcap_{n\in\mathbb{N}} f(A_n)$. So image of G_δ-sets under f are G_δ if f be open. However, feeble openness does not suffice to deduce that $\bigcap_{n\in\mathbb{N}} int(f(A_n)) \neq \emptyset$. Moreover, even if $\bigcap_{n\in\mathbb{N}} int(f(A_n))$ is non-void, then we can not imply that it is dense in Y.

Mirmostafaee and Piotrowski [7] proved that if f be quasi open and feebly continuous, and if U be an open dense subset of X, then $int(f(U))$ is dense in Y, but it is not enough to prove density of $\bigcap_{n\in\mathbb{N}} int(f(A_n))$, unless Y is a Baire space.

Suppose that X is the set of real numbers of with usual topology τ and τ' is the topology generated by $\tau \cup \{U \cap \mathbb{Q} : U \in \tau\}$. Then the identity map $i : (X, \tau') \to (X, \tau)$ is bijection, continuous and nearly feebly open function. Since every nonempty open subset of a Volterra space is weakly Volterra [1, Corollary 4.2], (X, τ') is not Volterra.

References

[1] D. B. Gauld, S. Greenwood and Z. Piotrowski, "On Volterra Spaces III: Topological Operations," Topol. Proc., vol. 23, pp. 167-182, 1998.

[2] D. Gauld and Z. Piotrowski, "On Volterra spaces," Far East J. Math. Sci., vol. 1, pp. 209-214, 1993.

[3] V. Volterra, "Alcune osservasioni sulle funzioni punteggiate discontinue," Giornale di Matematiche, vol. 19, pp. 76-86, 1881.

[4] J. Cao and H. J. K. Junnila, "When is a Volterra space Baire", Topol. Appl. vol. 154, pp. 527–532, 2007.

[5] S. Spadaro, "P-spaces and Volterra property", Bull. Aust. Math. Soc. vol 87, pp. 339–345, 2013.

[6] Z. Frolik, "Baire spaces and some generalizations of complete metric spaces," Czech. Math. J., vol. 11, pp. 237-248, 1961.

[7] A. K. Mirmostafaee and Z. Piotrowski, "On the dynamics of Baire and weakly Baire spaces," The 44^{th} Annual Iranian Mathematics Conference, 2013.

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Quotient and Product Riemannian Manifolds and There Metric Dimensions

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Abstract: Let (X, d) be a metric space. A subset A in X is called a resolver of (X, d), if each point x in X is uniquely determined by the distance $d(x, a)$ for each a in A. Also the metric dimension of (X, d) is the smallest integer $md(X, d) = md(X)$ such that there is a set A of cardinality $md(X)$ that resolves X. In general cases of the metric spaces we know very little about metric dimension, bout in the case that X is the vertex set of a graph, there are much investigations in this regards. In [1], the metric dimension of n-dimensional Euclidean space, Hyperbolic space, spherical space and some special subsets of them and some more is computed. In this work we are going to compute $md(X)$ for the case that X is a quotient Riemannian manifold or a product Riemannian manifold.

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Keywords quotient Riemannian manifold , product Riemannian manifold, metric space, resolving set, hyperbolic and spherical space,

Introduction

Let (X, d) be a metric space. A non-empty subset A of X is called a *resolver* of (X, d) if $d(x, a) = d(y, a)$ for all a in A implies $x = y$. The metric dimension $md(X)$ of (X, d) is the smallest integer k such that there is a resolver of (X, d) of cardinality k . A resolver of (X, d) with cardinality $md(X)$ is called a *metric basis* for X. As X resolves X every metric space X has a metric dimension which is at most the cardinality $|X|$ of X. For the first time the concept of the metric dimension of a metric space appeared in 1953 in [3], and attracted some more attention in 1975 when it was applied to the set of the vertices of a graph [7, 12]. Since then its applied in some more branches of the sciences and much has been published on this topic, see, for example [4, 5, 6, 8, 10]. Also in [1], $md(X)$ is computed in the cases that X is an n-dimensional Euclidian space E^n, spherical space S^n, Hyperbolic space H^n and open subsets and convex sets of them and also when X is a Riemann surface. In this paper we are going to compute the metric dimension of a quotient Riemannian manifold or a product Riemannian manifold.

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Notations and examples

Let (X, d) be a metric space. For two distinct points P and Q in X, we define the bisector $B(P|Q)$ by

$$B(P|Q) = \{x \in X \mid d(x, P) = d(x, Q)\} \quad (1)$$

A subset A of X fails to resolve X if and only if there are distinct points P and Q in X such that for all a in A,$d(P, a) = d(Q, a)$; then a subset A of X is a resolver of X if and only if it is not contained in any bisector.

For each open subset A in R (including R), since each point is between two points in A,(A is open) then $md(A) > 1$. Also for each two distinct point P and Q in A, $B(P|Q) = \{\frac{P+Q}{2}\}$ includes a single point. Then each subset including two distinct points in A is a resolver for A. Then $md(A) = 2$. But for $A = [a, b), a, b \in R$(including $b = \infty$),$\{a\}$ is a metric basis for A. Then $md(A) = 1$.

Example () shows that for $A \subseteq B \subseteq X$ in general non of the inequalities $md(A) \leq md(B)$ or $md(B) \leq md(A)$ is true. In fact the metric dimension of the subsets of a metric space depends hardly on the shape of the subset.

For each two distinct points P, Q in S^1 the unit circle, with $P \neq \pm Q$, $\{P, Q\}$ is a metric basis for S^1. Then $md(S^1) = 2$.

Let (X, d) be the following three standard n-dimensional geometries of constant curvature:

(1) *Euclidian space* E^n; that is $R^n = \{x = (x_1, ..., x_n) \mid x_i \in R\}$ with the metric $d(x, y) = \|x - y\|$.
(2) *Hyperbolic space* H^n; that is $H^n = \{x \in R^n \mid x_n > 0\}$ with path metric derived from $|dx|/x_n$.
(3) *Spherical space* S^n; that is $S^n = \{x \in R^{n+1} \mid \|x\| = 1\}$ with path metric induced by the Euclidian metric on R^{n+1}.

Then the metric dimension of (X, d) in each three cases is $n + 1$ [1].

We denote by $B_X(x, r)$, the open ball with center x and radius r in (X, d). When (X, d) is one of the above three cases, for each two distinct points P and Q, the bisector $B(P|Q)$ is a Euclidian, spherical or hyperbolic hyperplane [2, 9]. Then in these cases each subset of $n+1$ points in X that is not included in a hyperplane is a metric basis for X. Then, in these cases, each open ball $B_X(x, r)$ includes a metric basis for X. We need these notes and notations in the proofs of the next section.

1 The Metric Dimension of Quotient and Product Riemannian Manifolds

If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, the product $M_1 \times M_2$ has a natural Riemannian metric $g = g_1 \oplus g_2$, called the product metric, defined by

$$g(x_1 + x_2, y_1 + y_2) = g_1(x_1, y_1) + g_2(x_2, y_2).$$

where $x_i, y_i \in T_{p_i}M_i$ under the natural identification $T_{(p_1,p_2)}M_1 \times M_2 = T_{p_1}M_1 \oplus T_{p_2}M_2$.
When we have two metric spaces, in general case computing the metric dimension of the product metric space of them is very complicated and is not a constant function of there metric dimensions. But in the case of Riemannian manifolds(without boundary) we have the following theorem.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and d_i be the path metric derived from g_i. If the metric dimension of (M_i, d_i) be m_i, then the metric dimension of $(M_1 \times M_2, d)$ is $m_1 + m_2 - 1$. Where d is the path metric derived from the product metric $g = g_1 \oplus g_2$.

Sketch of the proof. We show that if $\{p_1, p_2, p_3, ..., p_{m_1}\}$ and $\{q_1, q_2, q_3, ..., q_{m_2}\}$ be metric basis respectively for (M_1, d_1) and (M_2, d_2), then $\{(p_1, q_i), (p_j, q_1) \mid i = 1, ..., m_1, j = 2, ..., m_2\}$ is a metric basis for $(M_1 \times M_2, d)$.

$md(R \times S) = 3$ and $md(T^n) = md(S^1 \times S^1 \times S^1 \times \cdots \times S^1) = 2n - (n-1) = n+1$.

Let Γ be a discrete group of isometries of an n-dimensional homogenous Riemannian manifold (X, g) acting freely on X. Then the orbit space X/Γ is called an *X-space-form*. By [11, Theorem 8.1.3], an X-space-form is an n-manifold.

Let $(X/\Gamma, \bar{d}_g)$ be an X-space-form equipped with the metric $\bar{d}_g$ that is induced by the path metric d_g derived from g, the metric of the homogenous Riemannian manifold (X, g). Then $md(X/\Gamma) = md(X)$.

Sketch of the proof. Let F be a fundamental domain for Γ in X. Since X is a homogenous Riemannian manifold one can show that F contains a metric basis. Let $\{p_1, p_2, p_3, ..., p_{md(X)}\}$ be a metric basis for (X, d_g) such that all p_i are in F. Then we show that $\{[p_1], [p_2], [p_3], ..., [p_{md(X)}]\}$ is a metric basis for $(X/\Gamma, \bar{d}_g)$.

$md(S^1) = md(R/Z) = 2$ and $md(RP^n) = md(S^n/Z_2) = n+1$.

References

[1] Bau, S., Beardon, A.F.: The metric dimension of metric spaces. Comput. Methods Funct. Theory 13, No. 2, 295-305 (2013).

[2] Beardon, A.F.: The Geometry of Discrete Groups. Springer, New York (1983)

[3] Blumenthal, L.M.: Theory and Applications of Distance Geometry. Clarendon Press, Oxford (1953)

[4] Boutin, D.L.: Determining sets, resolving sets, and the exchange property. Graphs Combin.25, 789806 (2009)

[5] Cceres, J., Hernando, C., Mora, M., Pelayo, I.M., Puertas, M.L.: On the metric dimension of infinite graphs. Electron. Notes Discrete Math.35, 1520 (2009)

[6] Chappell, G.C., Gimbel, J., Hartman, C.: Bounds on the metric and partition dimension of a graph. Ars Comb.88, 349366 (2008)

[7] Harary, F., Melter, R.A.: On the metric dimension of a graph. Ars Comb.2, 191195 (1976)

[8] Khuller, S., Raghavachari, B., Rosenfeld, A.: Landmarks in graphs. Discrete Appl. Anal.70, 217229 (1996)

[9] Matsuzaki, K. Taniguchi, M.: Hyperbolic manifolds and Kleinian groups. Oxford University Press, 1998.

[10] Melter, R.A., Tomescu, I.: Metric basis in digital geometry. Comput. Vis. Graph. Image Process.25,113121 (1984)

[11] Ratcliffe, J.G.: Foundations of Hyperbolic Manifolds. Springer, New York (1994)

[12] Slater, P.J.: Leaves of trees. Congr. Numer.14, 549559 (1975)

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Geometric Characterization of Riemannian Foliated Cocycles via Holonomy Groups

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Abstract: Foliations with transverse metrics or Riemannian foliated cocycles can be regarded as one the significant class of foliations mainly due to the fact that they are the natural setting to generalize Riemannian geometry to foliated manifolds. In this paper, taking into account the point that the holonomy group $\mathrm{Hol}(g)$ of a Riemannian manifold (M, g) demonstrates the geometrical structures on M compatible with g, we have comprehensively analyzed the Riemannian foliated cocycles via the holonomy groups. As a main consequence, a complete classification of this particular geometric structure is presented.

Keywords: Riemannian Foliated Cocycles, Holonomy Group, Riemannian Submersion, Transverse Bundle.

1 INTRODUCTION

In mathematics, foliation theory can be regarded as a powerful geometric device which is applied in order to study manifolds, consisting of an integrable subbundle of the tangent bundle. In other words, a foliation locally looks like a decomposition of the manifold as a union of parallel submanifolds of smaller dimension. Such foliations of manifolds occur naturally in various geometric fields such as solutions of differential equations and integrable systems or in differential topology. In fact, the concept of a foliation first appeared explicitly in the work of E.hresmann and Reeb in the 1940's [1], motivated by the question of existence of completely integrable vector fields on three-dimensional manifolds. Since that time, the subject has enjoyed a rapid undeniable development. Nowadays, foliations are the focus of a great deal of research activity in various fields.

In 1959 B.Reinhart introduced a particular type of foliations which is constructed via a particular geometric structure called Riemannian foliated cocycle [2]. When for a given foliation there exists a Riemannian metric g on M which is transverse for $\mathcal{F}$, we say that $(M, g, \mathcal{F})$ is a Riemannian foliated cocycle. This notion is quite intutive. As it will be demonstrated in the current paper, the existence of this geometric stucture leads to creation of a particular metric for which the leaves of the foliation remain locally at constant distance from each other. This class of foliations, can be regarded as a well candidate and a significant device for modeling situations drawn from mechanics and physics and particularly plays a fundamental role in generalizing Riemannian geometry to foliated manifolds. Consequently, Riemannian foliated cocycles form a natural class of foliations that is worth investigating from different aspects [3, 4, 5].

The holonomy group $\mathrm{Hol}(g)$ of a Riemannian manifold (M, g) is an area of Riemannian geometry which demonstrates the geometrical structures on

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M compatible with g. The investigation of Riemannian holonomy has led to a number of significant developments. The notion of holonomy was firstly introduced by Cartan (1926) in order to study and classify symmetric spaces [6]. It was not until much later that holonomy groups would be applied as a powerful tool in order to study Riemannian geometry in a more general setting. The decomposition and classification of Riemannian holonomy has important applications to physics, and particularly to string theory. In this paper, we have thoroughly investigated the Riemannian foliated cocycles via the holonomy groups. As a main consequence, some significant characterizations of this geometric structure is obtained via the mentioned approach.

2 Holonomy Groups and Riemannian Foliated Cocycles

Let M be a manifold, $\mathcal{P}$ a principal bundle over M with fibre G, and ∇ a connection on $\mathcal{P}$. For $p, q \in \mathcal{P}$, if there exists a piecewise smooth horizontal curve in $\mathcal{P}$ joining p to q, write $p \sim q$. It is clear that $\sim$ is an equivalence relation. For the fixed point $p \in \mathcal{P}$, the holonomy group of $(\mathcal{P}, \nabla)$ based at p is defined as follows:

$$\mathrm{Hol}_p(\mathcal{P}, \nabla) = \{g \in G : p \sim g.p\}$$

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇^l. Then the holonomy group $\mathrm{Hol}(g)$ of g can be defined as $\mathrm{Hol}(\nabla^l)$. Then $\mathrm{Hol}(g)$ is a subgroup of $\mathrm{O}(n)$, defined up to conjugation in $\mathrm{O}(n)$. We will refer to the holonomy group of a Riemannian metric as a Riemannian holonomy group.
Let M be a manifold of dimension n which is equipped with a foliation $\mathcal{F}$ of codimension q. The foliation $\mathcal{F}$ is called to be modeled on a q-dimensional manifold $\mathcal{M}_0^*$, if it is defined by a cocycle $\chi = \{U_i, \varphi_i, \psi_{ij}\}_I$ modeled on $\mathcal{M}_0^*$, i.e. the following conditions are satisfied [7]:
(a) : $\{U_i\}$ is an open covering of M.
(b) : $\varphi_i : U_i \longrightarrow \mathcal{M}_0^*$ are submersions with connected fibres, and
(c) : $\psi_{ij}\varphi_j = \varphi_i$ on $U_i \bigcap U_j$.
According to above definition, the q-manifold $\mathcal{M}^* = \bigsqcup \mathcal{M}_i^*$, $\mathcal{M}_i^* = \varphi_i(U_i)$, is denoted as the transverse manifold corresponding to the cocycle χ and the pseudogroup $\mathcal{H}$ of local diffeomorphisms of $\mathcal{M}^*$ generated by ψ_{ij} the holonomy pseudogroup representative on $\mathcal{M}^*$ which is associated to the cocycle χ. It is noticeable that $\mathcal{M}^*$ is a complete transverse manifold and the equivalence class of $\mathcal{H}$ is called the holonomy pseudogroup of $(M, \mathcal{F})$. In this paper, in what follows, it is assumed that $\mathcal{F}$ is constructed via a cocycle χ and the transverse manifold and holonomy pseudogroup associated to χ are denoted by $\mathcal{M}^*$ and $\mathcal{H}$, respectively. The foliation $\mathcal{F}$ is called (transversally) Riemannian, if it is constructed via a cocycle $\chi = \{U_i, \varphi_i, \psi_{ij}\}_I$ modeled on a Riemannian manifold $(\mathcal{M}_0^*, g_0)$ and such that the transformations ψ_{ij} are local isometries of the Riemannian structure g_0.
The concept of holonomy is closely related to that of a transverse Riemannian structure on the foliation. For a given point x on a manifold equipped with a foliation of codimension q, one may consider how the leaves near that point intersect a small q-dimensional disk which is transversal to the leaves and contains the given point. The ways in which these leaves depart from this disk and return to it are encoded in a group, called the holonomy group at x. It is a quotient group of the fundamental group of leaf through x. This group is of special importance, because it contains a lot of information about the structure of the foliation around the leaf through x.
A Riemannian metric g_Q on the transverse bundle $Q = \frac{TM}{T(F)}$ of a foliation F is holonomy invariant if $\mathcal{L}_X g_Q = 0$, for any $X \in \mathcal{X}(F)$.

Theorem 2.1. *Let M be a manifold of dimension n which is equipped with a foliation $\mathcal{F}$ of codimension q. Then $(M, g, \mathcal{F})$ is a Riemannian foliated cocycle if and only if the induced metric on the transverse bundle is holonomy invariant.*

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Proof: Let L_α be a leaf of $\mathcal{F}$, Υ a path in L_α, and let E and K be transversal sections of $\mathcal{F}$ with $\Upsilon(0) \in E$ and $\Upsilon(1) \in K$. Then we must prove that

$$\mathrm{Hol}^{E,K}(\Upsilon) : (E, \Upsilon(0)) \longrightarrow (K, \Upsilon(1))$$

is the germ of an isometry, in other words $\mathcal{H} = \mathrm{Hol}^{E,K}(\Upsilon)$ preserves the metric. According to the definition of holonomy, we can assume that Υ is inside a surjective chart, $\Omega = (x^1, ..., x^p, y^1, ..., y^q) : U \longrightarrow \mathcal{R}^p \times \mathcal{R}^q$ of $\mathcal{F}$ and $E, K \subset U$. Without loss of generality, assume that $\Omega(t) \subset \{0\} \times \mathcal{R}^q$, so that the vector fields $\frac{\partial}{\partial y^i}|_E$ form a frame for the tangent bundle of E. Furthermore assume that the holonomy diffeomorphism $\mathcal{H} : E \longrightarrow K$ is defined on all of E. By definition of H we have: $y^i o\mathcal{H} = y^i|_E, \quad for \quad i = 1, ..., q$. Therefore $\frac{\partial(y^i oh)}{\partial y^j}(p) = \delta_{ij}, \quad for \quad i, j = 1, ..., q$, so:

$$\mathcal{H}_\star(\frac{\partial}{\partial y^i}(p)) \in \frac{\partial}{\partial y^i}(\mathcal{H}(p)) + T_{\mathcal{H}(p)}(\mathcal{F}), \quad \forall p \in E.$$

Here we view $T_{\mathcal{H}(p)}(\mathcal{F})$ as a subspace of $T_{\mathcal{H}(p)}(M)$. Particularly we have [5]:

$$\begin{aligned} & g|_E(\mathcal{H}_{\star p}(\tfrac{\partial}{\partial y^i}(p)), \mathcal{H}_{\star p}(\tfrac{\partial}{\partial y^j}(p))) \\ & = g(\tfrac{\partial}{\partial y^i}(\mathcal{H}(p)), \tfrac{\partial}{\partial y^j}(\mathcal{H}(p))) \\ & = g_{ij}(\mathcal{H}(p)) = g_{ij}(p) = g|_E(\tfrac{\partial}{\partial y^i}(p), \tfrac{\partial}{\partial y^j}(p)). \end{aligned}$$

Consequently, for an arbitrary transversal section T at $x \in L_\alpha$ we obtain the map:

$$\mathrm{Hol}^E = \mathrm{Hol}^{E,E} : \pi_1(L_\alpha, x) \to Diff_x(E)$$

which is a group homomorphism from the fundamental group of the leaf L_α at x to the group of germs at x of local diffeomorphisms of E with respect to the point x. We will say that Hol^E is the holonomy representation of the leaf L_α at x. Its image is the holonomy group of L_α at x. $\diamondsuit$

Theorem 2.2. *Let M be a manifold of dimension n which is equipped with a foliation $\mathcal{F}$ of codimension q. Then $(M, g, \mathcal{F})$ is a Riemannian foliated cocycle if and only if for any $X, Y, Z \in \Gamma(TM)$ the following relation satisfied:*

$$\begin{aligned} & \pi X(g(\tilde{\pi}Y, \tilde{\pi}Z)) - g([\pi X, \tilde{\pi}Y], \tilde{\pi}Z) \\ & \qquad -g([\pi X, \tilde{\pi}Z], \tilde{\pi}Y) = 0. \end{aligned} \qquad (1)$$

Proof: Suppose that the tangent distribution $T(\mathcal{F})$ to the foliation is Riemannian. The complementary orthogonal distribution $T(\mathcal{F})^\perp$ to $T(\mathcal{F})$ in TM is Riemannian too, and we take it as transversal distribution to the folation $\mathcal{F}$. Also, we call $T(\mathcal{F})$ the structural distribution of $\mathcal{F}$. Here we denote by the same symbol g the Riemannian metrics induced by g on $T(\mathcal{F})$ and $T(\mathcal{F})^\perp$. The projection morphism of TM on $T(\mathcal{F})$ and $T(\mathcal{F})^\perp$ with respect to the decomposition: $TM = T(\mathcal{F}) \oplus T(\mathcal{F})^\perp$, are denoted by π and $\tilde{\pi}$ respectively. According to theorem (5.1) of [8], there exists a unique connection $\nabla^{\mathcal{I}}$ (resp. $\nabla^{\mathcal{I}^\perp}$) with respect to above decomposition. We call $\nabla^{\mathcal{I}}$ and $\nabla^{\mathcal{I}^\perp}$ the intrinsic connections on $T(\mathcal{F})$ and $T(\mathcal{F})^\perp$ respectively. According to [8], we have:

$$\begin{aligned} \nabla^{\mathcal{I}}{}_{\tilde{\pi}X}\pi Y &= \pi[\tilde{\pi}X, \pi Y], \\ \nabla^{\mathcal{I}^\perp}{}_{\pi X}\tilde{\pi}Y &= \tilde{\pi}[\pi X, \tilde{\pi}Y]. \end{aligned} \qquad (2)$$

Now, According to metric isomorphism $T(\mathcal{F})^\perp \approx \frac{TM}{T(\mathcal{F})}$, the proof completes. $\diamondsuit$

On a Riemannian manifold M, the distance between two points p and q can be defined as follows:

$$d(p, q) := inf\{L(\gamma) : \gamma : [a, b] \to M\}$$

where γ is a piecewise smooth curve with $\gamma(a) = p$ and $\gamma(b) = q$. We remark that any two points $p, q \in M$ can be connected by a piecewise smooth curve, and $d(p, q)$ therefore is always defined.
As a consequence of the above theorem, we have:

Theorem 2.3. *Let M be a manifold of dimension n which is equipped with a foliation $\mathcal{F}$ of codimension q. Then the following assertions are equivalent:*
(1): $(M, g, \mathcal{F})$ is a Riemannian foliated cocycle.
(2): The foliation $\mathcal{F}$ is locally defined by Riemannian submersions and the transverse changes of coordinates are isometries.
(3): The leaves of the foliation $\mathcal{F}$ locally remain at constant distance.

3 Conclusion

In this paper, we have presented a comprehensive investigation of the Riemannian foliated cocycles via the holonomy groups. The holonomy group of a Riemannian manifold can be regarded as one of the basic significant objects that one can associate to a Riemannian manifold. This geometric structure interacts with and contains fundamental information regarding to a great number of geometric properties of the manifold. In this paper, we have demonstrated that the existence of a Riemannian foliated cocycle on a Riemannian manifold is equivalent to the existence of a Riemannian metric for which the length of tangent vectors or curves which are transversal to the leaves is invariant under the holonomy group. In this case, the holonomy group can be interpreted as a group of isometries of a transversal disk. Furthermore, some geometric characterization of Riemannian foliated cocycles is presented via the mentioned approach.

References

[1] Ch.Ehresmann, G.Reeb, Sur les champs d'*éléments* de contact de dimension p compl*é*tement int*é*grables dans une vari*été* continuement diff*é*rentiable V_n, *C.R.Acad.Sci.Paris*, **216** pp. 628-630, 1944.

[2] B.L.Reinhart, "Foliated manifolds with bundle-like metric", Annals of math **69(2)**, pp. 119-132, 1959.

[3] P.Molino, "Riemannian Foliations", Progress in Math , **73**, Birkhauser, Boston, 1988.

[4] Ph. Tondeur, "Geometry of Foliations", Birkhauser, Basel, 1997.

[5] I. Moerdijk, J. Mrcun, "Introduction to Foliations and Lie Groupoids", Cambridge University Press, 2003.

[6] D. D.Joyce, "Riemannian Holonomy Groups and Calibrated Geometry", Oxford university press, New York, 2007.

[7] M. Józefowicz, R. Wolak, "Finslerian foliations of compact manifolds are Riemannian", Differential Geometry and its Applications, Vol. 26 (2), pp. 224-226, April 2008.

[8] A. Bejancu, H. Reda.Farran , "Foliations and Geometric Structures", Springer-Verlag, 2006.

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A note on generalized ϕ-recurrent Sasakian manifolds

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Abstract: In this paper we study ϕ-recurrent and generalized ϕ-recurrent Sasakian manifolds. We prove that ϕ-recurrent Sasakian manifolds are ϕ-symmetric and a generalized ϕ-recurrent Sasakian manifold has constant curvature.

Keywords: Generalized ϕ-recurrent manifolds, contact manifold, constant curvature.

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1 INTRODUCTION

It is well known that almost contact manifolds are classified by the sign of sectional curvature (c) of the plane section that contains the associated vector field ξ [1]. Here we study the case in which $c > 0$ and is called Sasakian manifold. Sasakian manifolds are the normal contact metric manifolds and these are the odd-dimensional analogs of kähler manifolds.

Takahashi [8] introduced the notion of locally ϕ-symmetric sasakian manifold. As its generalization, ϕ-recurrent Sasakian manifolds and generalized ϕ-recurrent Sasakian manifolds are defined and studied in many papers [2, 3, 4, 6]. In this paper we investigate ϕ-recurrent and generalized ϕ-recurrent Sasakian manifolds. We show thata ϕ-recurrent Sasakian manifold is ϕ-symmetric. This means that there dose not exist any none trivial ϕ-recurrent Sasakian manifold. Moreover, here it has been shown that generalized ϕ-recurrent Sasakian manifolds are of constant curvature and so they are locally symmetric and ϕ-symmetric.

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2 ϕ-RECURRENT SASAKIAN MANIFOLDS

Let (M, ϕ, ξ, η, g) be a $2n+1$ dimensional almost contact metric manifold, where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric which satisfy the following conditions for any $X \in \mathcal{T}(M)$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1 \tag{1}$$

$$\phi^2 X = -X + \eta(X)\xi \ , \ g(\xi, X) = \eta(X). \tag{2}$$

If,

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{3}$$

and

$$(\nabla_X \xi) = X - \eta(X)\xi, \tag{4}$$

then (M, ϕ, ξ, η, g) is called a Sasakian manifold [1]. In a Sasakian manifold the following relations hold

$$(\nabla_X \xi) = -\phi(X), \quad (\nabla_X \eta)Y = g(X, \phi Y), \tag{5}$$

$$R(X, Y, \xi) = -\eta(X)Y + \eta(Y)X, \tag{6}$$

$$S(X, \xi) = 2n\eta(X), \tag{7}$$

$$(\nabla_W R)(X, Y)\xi = -g(\phi X, W)Y + g(W, \phi Y)X - R(X, Y)\phi W. \tag{8}$$

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for any $X, Y, W \in \mathcal{T}(\mathrm{M})$, where R is the Riemannian curvature tensor and S is the Ricci tensor.

Definition 2.1. *A Sasakian manifold is called ϕ-symmetric [8] if its curvature tensor R satisfies*

$$\phi^2(\nabla_W R)(X, Y, Z) = 0 \tag{9}$$

for any vector fields X, Y, Z and W in $\mathcal{T}(M)$.

Definition 2.2. *A Sasakian manifold is called ϕ-recurrent [5] if there exists a 1-form A such that the curvature tensor R satisfies*

$$\phi^2(\nabla_W R)(X, Y, Z) = A(W)R(X, Y)Z, \tag{10}$$

and a generalized ϕ-recurrent [7] if there exist 1-forms A and B such that

$$\phi^2(\nabla_W R)(X, Y, Z) = A(W)R(X, Y)Z + B(W)(g(Y, Z)X - g(X, Z)Y),$$

for any vector fields X, Y, Z and W in $\mathcal{T}(M)$.

Definition 2.3. *A Riemannian manifold (M, g) is called Ricci symmetric if its Ricci tensor S satisfies the condition*

$$(\nabla_W S)(X, Y) = 0, \tag{12}$$

for any vector fields X, Y and W in $\mathcal{T}(M)$, and it is called Ricci ϕ-symmetric [5] if

$$\phi^2(\nabla_W S)(X, Y) = 0. \tag{13}$$

2.1 MAIN RESULTS

Theorem 2.4. *(M^{2n+1}, g) is a ϕ- recurrent Sasakian manifold if and only if it is ϕ- symmetric.*

Proof. Using (2) and (9) and contracting the equations, we have

$$-\nabla_W S(Y, Z) + g((\nabla_W R)(\xi, Y, Z), \xi) = A(W)S(Y, Z). \tag{14}$$

Replacing Z and Y by ξ implies

$$-2n\eta(\phi W) = -2nA(W). \tag{15}$$

Thus by (1) we have

$$2nA(W) = 0. \tag{16}$$

Since $n \neq 0$, we get $A(W) = 0$ and

$$\phi^2((\nabla_W R)(X, Y, Z)) = 0. \tag{17}$$

The converse is clear by putting $A = 0$. □

Proposition 2.5. *A ϕ- recurrent Sasakian manifold (M^{2n+1}, g) is a Ricci symmetric manifold.*

Proof. Because of (14) M is a Ricci recurrent manifold. Since $A(W) = 0$, M is a Ricci symmetric manifold. □

It has been proved [6] that the 1-forms A and B are related as $A = -B$ in generalized ϕ-recurrent Sasakian manifolds, so we can say that a Sasakian manifold is generalized ϕ-recurrent if and only if

$$\phi^2(\nabla_W R)(X, Y, Z) = A(W)[R(X, Y)Z - g(Y, Z)X + g(X, Z)Y].$$

Theorem 2.6. *Any generalized ϕ- recurrent Sasakian manifold, has constant sectional curvature.*

Proof. By using Definition 2.2, (2) and (6) we have

$$-(\nabla_W R)(X, Y)\xi = A(W)[R(X, Y)\xi + \eta(Y)X - \eta(X)Y] = 0. \tag{18}$$

Now, in view of (9) and (19), we get

$$-g(\phi X, W)Y + g(W, \phi Y)X - R(X, Y)\phi W = 0. \tag{19}$$

From [5], we know

$$R(X, Y)\phi W = \phi R(X, Y)W + g(\phi X, W)Y - g(W, \phi Y)X - g(Y, Z)\phi X + g(X, Z)\phi Y. \tag{20}$$

So, (19) and (21) imply

$$\phi R(X, Y)W + g(X, W)\phi Y - g(W, Y)\phi X = 0. \tag{21}$$

Now, applying ϕ on (21) and using (3) and (6) imlpy

$$R(X, Y)W = g(X, W)Y - g(W, Y)X, \tag{22}$$

for any X,Y and W. This means that M is of constant curvature 1. □

Since the curvature is constant, we can conclude the following.

Corollary 2.7. *A generalized ϕ-recurrent Sasakian manifold is locally symmetric.*

References

[1] D. E. Blair. *Riemannian geometry of contact and sympelectic manifolds.* progress mathematics 203, Brikhauser, Boston-Basel-Berlin, 2002.

[2] D. E. Blair, J. Kim, M. Tripathi. *On the concircular curvature tensor of a contact metric manifold.* J. Korean Math. Soc. 42(5) , pp 883-893, 2005.

[3] E. Boeckx. *A class of locally ϕ-Symmetric contact metric spaces*, Arch. Math. vol. 72 ,pp 466-472, 1999.

[4] E. Boeckx, P. Buekeny and L. Vanhecke. *ϕ-Symmetric contact metric spaces*, Glasgow Math. J. vol. 41, pp 409-416, 1999.

[5] U. C. De, A. A. Shaikh, Sudipta Biswas. *On Φ-recurrent Sasakian manifolds*, Novi Sad J. Math. vol 33, pp 43-48, 2003.

[6] J. Kim, R. Prasad, M. Tripathi. *On generalized Ricci-recurrent trans-Sasakian manifolds.* J. Korean Math. Soc. vol 39, pp 953-961, 2002.

[7] D. A. Patil, D. G. Prakasha, C. S. Bagewadiu.*On generalized ϕ-recurrent Sasakian manifolds*,Bulletin of Mathematical Analysis and Applications. 1(3) , pp 42-48, 2009.

[8] T. Takahashi. *Sasakian ϕ-symmetry spaces*, Thoku Math. J. vol. 29, pp 91-113 1977.

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Classifying the differential invariants of Lie symmetry groups of Korteweg-deVries equation by the equivariant moving frame method

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Abstract: The aim of this paper is classifying the differential invariants of Lie symmetry groups of Korteweg-deVries (KdV) equation and finding the structure of the induced differential invariant algebra, based on the method of equivariant moving frames.
Keywords: KdV equation, equivariant moving frame, differential invariant,recurrence formulas, differential invariant algebra.

1 INTRODUCTION

In recently years equivariant moving frame method has been many promotions, can be found details in[1] and [5].

P.J. Olver [4] explain how can obtain recurrence formula and structure of algebras of differential invariants for finite-dimensional Lie groups with equivariant moving frame method. E.L. Mansfeld [6] has obtained algebras of differential invariants with out calculate Murrer - Cartan forms and only with correction matrix, **K** and whole of computations has briefed in matrix computations.

In this article we explain method of obtaining recurrence formula and structure of algebras of differential invariants with correction matrix, **K**; on the other hand **K** matrix components are generators of differential invariants algebras so we can classify solutions of KdV equation as surfaces under symmetry groups of this equation.

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2 Korteweg-deVries Equation and Their Symmetry Groups

We celebrated Korteweg-deVries (KdV) equation,

$$u_t + u_{xxx} + uu_x = 0 \tag{1}$$

The total space $M = R^3$ has coordinates (t, x, u), and its solutions $u = f(t, x)$ define $p = 2$ dimensional submanifolds of M. We want to consider KdV equation and their symmetry groups, for more details see [8],

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$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial t},$$

$$\mathbf{v}_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \mathbf{v}_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

3 Equivariant Moving Frame

We start by explanation the equivariant moving frame construction. Let G be an r-dimensional Lie group acting smoothly on an m-dimensional manifold M.

Definition 3.1. *([5]) A moving frame is a smooth, G-equivariant map $\rho : M \longrightarrow G$. There are two principal types of equivariance:*

$$\rho\,(g \cdot z) = \begin{cases} g \cdot \rho\,(z) & \text{left moving frame} \\ \rho\,(z) \cdot (g)^{-1} & \text{right moving frame} \end{cases}$$

Theorem 3.2. *([5]) Moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

For existance of moving frame, G must be act free and regular on M. Acting freely meaning that isotropy subgroup $G_x = \{g \in G \mid g \cdot x = x\} = \{e\}$ or equivalently, the orbits all have the same dimension, r, as G itself. Regularity requires that, in addition, the orbits form a regular foliation. In practice these conditions satisfy respectively by prolongation of group action sufficently larg and by choice of a appropriate cross section.

Theorem 3.3. *([5]) Let G act freely and regularly on M, and let $\mathcal{K} \subset M$ be a cross section. Given $z \in M$, let $g = ?(z)$ be the unique group element that maps z to the cross section: $g \cdot z = ?(z) \cdot z \in \mathcal{K}$. Then $\rho : M \longrightarrow G$ is a right moving frame.*

In aplication, we use the following procedure. We define the cross section $\mathcal{K}$ as a set of equations $\psi_k(z) = 0$, $k = 1, \cdots, r$. The number of equations, r, equals the dimension of the group. In order to $\mathcal{K}$ obtain the group element that takes z to $\mathcal{K}$, we solve the so called normalisation equations, $\psi_k(z) = 0, k = 1, \cdots, r$.

The frame $\rho(z)$ therefore satifies $\psi_i(\rho(z) * z) = 0, i = 1, \cdots, r$. If the solution is unique on the domain U, then ρ is a right frame, one that satisfies $\rho(g * z) = \rho(z) \cdot g^{-1}$. One chooses the normalisation equations to minimise the computations as much as possible for the application at hand. ([6])

3.1 A moving frame for KdV equation

We compute moving frame by the explaine method in pre-section for KdV:

$$a = (3tu - x)u_x^{1/3}, \qquad b = -tu_x,$$
$$c = -u/u_x^{2/3}, \qquad d = 1/3\ln(u_x).$$

4 Invariantisation map, Reccurence Formulas and Differential Invariant Algebra

4.1 Invariantisation map

Definition 4.1. *([6]) The map $z \longrightarrow I(z) = \rho(z) \cdot z$ is called the invariantisation map.*

Definition 4.2. *([6]) For any prolonged action in $(x_i, u^\alpha, u^\alpha_k)$-space, the specific components of $I(z)$, the invariantised jet coordinates, are denoted*

$$J_i = I\,(x_i)\,, \qquad I^\alpha = I\,(u^\alpha)\,, \qquad I^\alpha_k = I\,(u^\alpha_k)\,.$$

Definition 4.3. *([6]) We define total differential operator:*

$$D_i = \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \sum_k u^\alpha_{ki} \frac{\partial}{\partial u^\alpha_k}$$

that is an operator on the jet space of a manifold.

Definition 4.4. *([6]) Also we define:*

$$\tilde{D}_i = \frac{D}{D\tilde{x}_i} = \sum_{k=1}^{p} \left(\tilde{D}x\right)_{ik} D_k$$

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such that:

$$\left(\tilde{D}x\right)=\left(\left(D\tilde{x}\right)^{-1}\right)_{ik},D\tilde{x}=\begin{pmatrix}\frac{\partial\tilde{x}_1}{\partial x_1}&\cdots&\frac{\partial\tilde{x}_1}{\partial x_p}\\ \vdots&\ddots&\vdots\\ \frac{\partial\tilde{x}_p}{\partial x_1}&\cdots&\frac{\partial\tilde{x}_p}{\partial x_p}\end{pmatrix}$$

Definition 4.5. *([6]) A set of distinguished invariant operators is defined by evaluating the transformed total differential operators on the frame. They are $\mathcal{D}_j=\tilde{D}_j\mid_{g=\rho(z)}$ where the $\tilde{D}_j$ are given in definition 4.4.*

4.2 Recurrence Formulas

Definition 4.6. *([6]) The correction terms N_{ij},M^{α}_{kj} are defined by*

$$\mathcal{D}_jJ_i=\delta_{ij}+N_{ij},\qquad \mathcal{D}_jI^{\alpha}_k=I^{\alpha}_{kj}+M^{\alpha}_{kj},\quad (2)$$

where δ_{ij} is the Kronecker delta.

Definition 4.7. *([6]) For $\xi^j_i=\xi^j_i\left(x,u^{\beta}\right)$ we define: $\xi^j_i\left(I\right)=\xi^j_i\left(J,I^{\beta}\right)$, also similarly for $\phi^j_i\left(I\right)=\phi^j_i\left(J,I^{\beta}\right)$.*

Theorem 4.8. *([6]) There exists a $p\times r$ correction matrix $\mathbf{K}$ such that*

$$N_{kj}=\sum_{l=1}^{r}\mathbf{K}_{jl}\xi^k_l\left(I\right),\qquad M^{\alpha}_{kj}=\sum_{l=1}^{r}\mathbf{K}_{jl}\phi^{\alpha}_{k,l}\left(I\right).\quad (3)$$

Theorem 4.9. *([6]) The correction matrix $\mathbf{K}$, which provides the error terms in the process of invariant differentiation in 3 is given by*

$$\mathbf{K}=-TJ\left(\Phi J\right)^{-1},\quad (4)$$

where T,J and Φ are defined below:

Suppose the n variables actually occurring in the $\psi_{\lambda}\left(z\right)$ are $\zeta_1,\cdots,\zeta_n$; typically m of these will be independent variables and $n-m$ of them will be dependent variables and their derivatives. Define T to be the invariant $p\times n$ total derivative matrix

$$T_{ij}=I\left(\frac{D}{Dx_i}\zeta_j\right),$$

Also, let Φ denote the $r\times n$ matrix of infinitesimals with invariantised arguments,

$$\Phi_{ij}=\left(\frac{\partial\left(g\cdot\zeta_j\right)}{\partial g_i}\right)\Big|_{g=e}\left(I\right),$$

Furthermore, define J to be the $n\times r$ transpose of the Jacobian matrix of the left hand sides of the normalisation equations $\psi_1,\ldots,\psi_r$ with invariantised arguments, that is

$$J_{ij}=\frac{\partial\psi_j\left(I\right)}{\partial I\left(\zeta_i\right)}.$$

4.3 Differential Invariant Algebras

Definition 4.10. *([6]) We denote by $\mathcal{I}^0$ the set of zeroth invariants,*

$$\mathcal{I}^0=\{I\left(x_j\right)=J_i,I\left(u^{\alpha}\right)=I^{\alpha}\mid j=1,\ldots,p,\alpha=1,\ldots,q\}.$$

Theorem 4.11. *([6]) Suppose the normalisation equations $\psi_k=0$, $k=1,\ldots r$ yield a frame for a regular free action on some open set of the prolonged space with coordinates $(x_j,u^{\alpha},u^{\alpha}_k)$. Then the components of the correction matrix $\mathbf{K}$, given in Theorem 4.8, together with $\mathcal{I}^0$, given in definition 4.10, form a generating set of differential invariants.*

5 Differential Invariants and Recurrence Formulas for the KdV Equation

Infinitesimals table leads to:

	x	t	u	u_x	u_t	u_{xx}	u_{xt}	u_{tt}
a	1	0	0	0	0	0	0	0
b	0	1	0	0	0	0	0	0
c	t	0	1	0	$-u_x$	$-2u_{xx}$	$-3u_{xt}$	$-2u_{xt}$
d	x	$3t$	$-2u$	$-3u_x$	$-5u_t$	$-4u_{xx}$	$-6u_{xt}$	$-8u_{tt}$

By considering normalization equations we have:

$$\tilde{x}=0, \qquad \tilde{t}=0, \qquad \tilde{u}=0, \qquad \tilde{u}_x=1,$$
$$(\psi_1,\psi_2,\psi_3,\psi_4)(x,t,u,u_x)=(x,t,u,u_x-1),$$

and

$$\zeta_1=x, \quad \zeta_2=t, \quad \zeta_3=u, \quad \zeta_4=u_x,$$

$$J^x=0, \quad J^t=0, \quad I^u=0, \quad I^u_1=1,$$

so we find **K** matrix:

$$\Phi=\begin{array}{c} \\ a\\ b\\ c\\ d\end{array}\begin{array}{c}\begin{array}{cccc} x & t & u & u_x\end{array}\\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{pmatrix}\end{array},$$

$$J=\begin{array}{c} \\ J^x\\ J^t\\ I^u_1\\ I^u\end{array}\begin{array}{c}\begin{array}{cccc}\psi_1(I) & \psi_2(I) & \psi_3(I) & \psi_4(I)\end{array}\\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\end{array},$$

$$T=\begin{array}{c} \\ x\\ t\end{array}\begin{array}{c}\begin{array}{cccc} x & t & u & u_x\end{array}\\ \begin{pmatrix}1&0&1&I^u_{11}\\0&1&I^u_2&I^u_{12}\end{pmatrix}\end{array},$$

$$-K=\begin{array}{c} \\ x\\ t\end{array}\begin{array}{c}\begin{array}{cccc} a & b & c & d\end{array}\\ \begin{pmatrix}1&0&1&\frac{-I^u_{11}}{3}\\0&1&I^u_2&\frac{-I^u_2}{3}\end{pmatrix}\end{array}.$$

Therefore with **K** matrix and definition 4.6 we can achieve recurrence terms:

$$\mathcal{D}_x I^u_2 = I^u_{12} - 5I^u_2 I_{11} + 1,$$
$$\mathcal{D}_x I^u_{11} = I^u_{111} - \frac{4}{3}(I^u_{11})^2,$$
$$\mathcal{D}_x I^u_{12} = I^u_{122} + 2I^u_{12} - \frac{8}{3}I^u_{11}I^u_{22},$$
$$\mathcal{D}_t I^u_2 = I^u_{22} + I^u_2 - \frac{5}{3}I^u_{12}I^u_2,,$$
$$\mathcal{D}_t I^u_{22} = I^u_{222} + I^u_{12}(2I^u_2 - \frac{8}{3}I^u_{22}),$$
$$\vdots$$
$$\Downarrow$$

$$I^u_{12} = \mathcal{D}_x I^u_2 + 5I^u_2 I_{11} - 1,$$
$$I^u_{111} = \mathcal{D}_x I^u_{11} + \frac{4}{3}(I^u_{11})^2,$$
$$I^u_{112} = \mathcal{D}_x I^u_{12} - 2I^u_{12} + \frac{8}{3}I^u_{11}I^u_{22},$$
$$I^u_{22} = \mathcal{D}_t I^u_2 - I^u_2 + \frac{5}{3}I^u_{12}I^u_2,$$
$$I^u_{222} = \mathcal{D}_t I^u_{22} - I^u_{12}(2I^u_2 - \frac{8}{3}I^u_{22}),$$
$$\vdots$$

So according to theorem 4.11 differential invariant generators for algebra of the action are:

$$I^u_2 = \frac{uu_x + u_t}{u_x^{5/3}}, \qquad I^u_{11} = u_x^{-4/3},$$

and also differential invariant operators are:

$$\mathcal{D}_x = \frac{1-u}{u_x^{2/3}}D_x, \qquad \mathcal{D}_t = \frac{1}{u_x}D_t.$$

References

[1] M.Fels and P.J.Olver, *Moving coframes. II. Regularization and theoretical foundations*, Acta Appl. Math.55 (1999), 127-208.

[2] P.J. Olver, *Equivalence, invariant and symmetry*, Cambridge University Press, Cambridge (1995).

[3] P.J. Olver, *Applications of Lie groups to differential equations*, Springer Verlag, Second Edition, GTM, Vol. 107, New York, (1993).

[4] P.J.Olver, *Differential Invariant Algebra*, Comtemp. Math. 549 (2011), 95-121.

[5] P.J.Olver, *Lectures on Moving frames*, London Math. Soc. Lecture Note Series, vol. 381? Cambridge University Press? Cambridge, 2011, pp.207-246.

[6] E.L. Mansfeld, *A Practical Guide to the Invariant Calculus*, Cambridge University Press, Cambridge 2010.

[7] N.H.Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*,

[8] J.Cheh,P.J.Olver and J.Pohjanpelto, *Algorithms for Differential Invariants of Symmetry Groups of Differential Equations*, Found. Comput. Math. 8 (2008), 501-532.

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تعمیم شرط $\Delta x = \lambda x$ در قضیه تاکاهاشی به ابررویه‌های فضاگون در فضای دسیتر

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چکیده: مطالعه زیر منیفلدها و ابررویه‌های مینیمال و نیز ابررویه‌های با انحنای میانگین ثابت در فضای اقلیدسی و کره‌ی اقلیدسی توسط تاکاهاشی [۶] آغاز شد و پس از چند مرحله تعمیم و تحول به صورت مسئله شناسایی ابررویه‌هایی در آمد که میدان برداری موضع آنها در شرط $\Delta x = Ax + b$ صدق می‌کند که در آن A یک ماتریس مربعی ثابت و b یک بردار ثابت و Δ عملگر لاپلاس می‌باشد ([۴]) . سپس نتایج این مسئله به ابررویه‌های فروبرده شده در فضافرم‌های استاندارد ریمانی توسعه داده شد ([۲]). همچنین لوئیس جی .الیاس و دیگران در مرجع [۱] مسئله را به زیر منیفلدهای فضاهای شبه اقلیدسی گسترش دادند. اخیراً در مقاله مشترک الیاس و کاشانی ([۳]) شرط مذکور به صورت $L_k x = Ax + b$ روی ابررویه‌های فضافرم‌های ریمانی استاندارد (فضای اقلیدسی ، کره و فضای هذلولوی) مطرح شده و ابررویه‌های مورد نظر رده بندی شده است که A یک ماتریس خودالحاق و b یک بردار است و L_k عملگر خطی وابسته اولین وردش انحنای میانگین ابررویه است. در واقع L_k یک تعمیم از مرتبه k برای عملگر لاپلاس به شمار می‌رود. در این مقاله مسئله شناسایی ابررویه‌های فضاگون در شبه کره ی لورنتزی صادق در شرط $L_k x = Ax + b$ مورد نظر است .

کلمات کلیدی: ابررویه‌ی (لورنتزی ، فضاگون)، k -مین انحنای میانگین،k -مینیمال، عملگر خطی شده.

مقدمه

فضای برداری $\mathbb{R}^m$ با ضرب عددی (یا متریک) $< x,y >:= -\Sigma_{i=1}^{q} x_i y_i + \Sigma_{i>q} x_i y_i$ را فضای (شبه -) اقلیدسی نامیده و با $\mathbb{R}^m_q$ نمایش داده می‌شود که در آن $0 \leq q < m$. $\mathbb{R}^m_1$ فضای مینکوفسکی است.برای عدد حقیقی مثبت r و $q = 0,1$ ، $\mathbb{S}^{n+1}_q(r) = \{y \in \mathbb{R}^{n+2}_q \mid < y,y >= r^2\}$ نمایش دهنده ی (شبه -) کره یا فضای دسیتر با شعاع r و انحنای $\frac{1}{r^2}$ و $\mathbb{H}^{n+1}_q(-r) = \{y \in \mathbb{R}^{n+2}_{q+1} \mid < y,y >= -r^2\}$ نمایش دهنده‌ی (شبه)-هذلولوی یا فضای آنتی دسیتر با شعاع r و انحنای $-1/r^2$ است. فضافرم همبند ساده‌ی $M^{n+1}_q(c)$ با انحنای c و اندیس q برای $c = 0$، $\mathbb{R}^{n+1}_q$، برای $c = 1$، $\mathbb{S}^{n+1}_q = \mathbb{S}^{n+1}_q(1)$ است (با متریک القایی از $\mathbb{R}^{n+2}_{q+1}$). وقتی $q = 0$، مولفه‌ای از $\mathbb{H}^{n+1}_1$ را در نظر می‌گیریم.بر ابررویه فضاگون $x : M^n_p \to \tilde{M}^{n+1}_q(c)$ که $(0 \leq p \leq q \leq 1)$ در فضافرم ریمانی یا لورنتزی $\tilde{M}^{n+1}_q$ با $(q = 0,1)$، هر میدان برداری یکه مماس را که در هر نقطه از M یک بردار ویژه‌ی عملگر شکل S وابسته به یک میدان برداری یکه ی قائم N باشد، یک جهت اصلی و مقدار ویژه متناظر را یک انحنای اصلی M نامند. هر گاه S بر M (یا زیر مجموعه بازی از آن) قطری شدنی و

$\Delta x = Ax + b$ باشد. آنگاه، M_p^n یک ابررویه مینیمال یا هم پارامتر است.

قضیه ۴. ([۳]) فرض کنید $x : M^n \to \tilde{M}^{n+1}(c) \subset \mathbb{R}_q^{n+2}$ یک ابررویه جهت پذیر فرو برده شده بتوی کره ی $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ باشد (اگر $c = 1$). آنگاه x در شرط $L_k x = Ax$ برای یک ماتریس خود الحاق A، صدق می‌کند اگر و تنها اگر M یکی از ابر رویه های پایین باشد:

(آ) یک ابرریه با H_k ثابت و $H_{k+1} = 0$ ؛

(ب) یک زیر مجموعه باز از حاصلضرب ریمانی استاندارد $\mathbb{S}^m(\sqrt{1-r^2}) \times \mathbb{S}^{n-m}(r)$، که $0 < r < 1$ و $0 < m < n$، اگر $c = 1$.

قضیه ۵. ([۳]) فرض کنید $x : M^n \to \tilde{M}^{n+1}(c) \subset \mathbb{R}_q^{n+2}$ یک ابررویه جهت پذیر فروبرده شده در $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ (اگر $c = 1$) بوده و بر آن H_k ثابت باشد. آنگاه x در شرط $L_k x = Ax + b$ برای ماتریس خودالحاق $A \in \mathbb{R}^{(n+2)\times(n+2)}$ و یک بردار ناصفر $b \in \mathbb{R}^{n+1}$، صدق می‌کند اگر و تنها اگر :

(۱) $c = 1$ و M زیر مجموعه بازی از یک کره تماماً نافی $\mathbb{S}^n(r) \subset \mathbb{S}^{n+1}$ است که $0 < r < 1$.

(آ) زیر مجموعه بازی از یک کره $\mathbb{S}^n(r)$ که $r > 0$ ،

(ب) یک زیر مجموعه‌ی باز از یک فضای اقلیدسی $\mathbb{R}^n$.

قضیه ۶. ([۵]) فرض کنید M^n یک ابررویه فضاگون جهت پذیر همبند باشد که با فروبری طولپای $x : M^n \to \mathbb{S}_q^{n+1} \subset \mathbb{R}_q^{n+2}$ در (شبه -) کره ی $\mathbb{S}_q^{n+1}$ نگاشته شده است که در آن $q = 0, 1$ و H_k بر M ثابت است آنگاه، X در شرط $L_k x = Ax + b$ برای یک ماتریس حقیقی A ، و بردار حقیقی b و یک عدد صحیح نامنفی $k < n$ ، صدق می‌کند اگر و تنها اگر M زیر مجموعه بازی از یکی از ابررویه های پایین باشد :

(آ) $\mathbb{S}^n(r)$ که $0 < r < 1$ است اگر $q = 0$ و $r \le 1$ اگر $q = 1$ ؛

(ب) $\mathbb{R}^n$ یا $\mathbb{H}^n(-r)$ اگر $q = 1$.

قضیه ۷. ([۵]) فرض کنید M^n یک ابررویه فضاگون جهت پذیر همبند باشد که با فروبری طولپای $x : M^n \to \mathbb{S}_q^{n+1} \subset \mathbb{R}_q^{n+2}$ در (شبه -) کره ی $\mathbb{S}_q^{n+1}$ نگاشته شده است که در آن $q = 0, 1$.

$k_1, ..., k_n$ انحناهای اصلی M باشد، برای هر عدد صحیح j ($1 \le j \le n$) ، j-مین انحنای میانگین M را با H_j نمایش داده و با فرمول $\binom{n}{j} = (-\epsilon)^j s_j$ تعریف می‌شود که در آن $\epsilon = - < N, N >$ و $s_j := \Sigma_{1 \le i_1 < ... < i_j \le n} k_{i_1} ... k_{i_j}$.

ابررویه فضاگون $x : M^n \to \mathbb{R}_q^{n+1}$ را j-مینیمال گویند اگر H_{j+1} بر M متحد با صفر باشد.

عملگر خطی شده حاصل از وردش اول $(j+1)$-مین انحنای میانگین ابررویه ی $x : M_p^n \to \tilde{M}_q^{n+1}(c)$ و $0 \le p \le q \le 1$) عبارت است از نگاشت هموار $L_j : C^\infty(M) \to C^\infty(M)$ با ضابطه ی $L_j(f) := tr(P_j \circ \nabla^2 f)$, که در آن $\nabla^2 f$ به‌صورت $< \nabla^2 f(X), Y >= Hess(f)(X, Y)$ برای هر دو میدان برداری $X, Y \in \chi(M)$ به‌دست می‌آید که در اینجا عملگر $P_j : \chi(M) \to \chi(M)$، مرتبط با عملگر شکل S از M زمانی که I بر $\chi(M)$ همانی است، به شیوه ی استقرائی

$$P_0 = I, \quad P_j(-\epsilon)^j s_j I + \epsilon S \circ P_{j-1}, \; j = 1, \cdots, n,$$

تعریف می‌شود.

نتایج

در این بخش نتایج اصلی را به شکل لم‌ها و قضایا بیان می‌کنیم:

لم ۱. ([۲]) فرض کنید که $x : M_p^n \to \mathbb{S}_q^{n+1}$ ($0 \le p \le q \le 1$) یک ابررویه باشد که در شرط $\Delta x = Ax + b$ صدق می‌کند. اگر M_p^n دارای انحنای میانگین غیر ثابت باشد آنگاه، $b = 0$ است.

گزاره ۲. ([۲]) فرض کنید که $x : M_p^n \to \mathbb{S}_q^{n+1}$ ($0 \le p \le q \le 1$) یک فروبری طولپا صادق در شرط $\Delta x = Ax + b$ باشد. آنگاه، M_p^n دارای انحنای میانگین ثابت است.

قضیه ۳. ([۲]) فرض کنید که $x : M_p^n \to \mathbb{S}_q^{n+1}$ ($0 \le p \le q \le 1$) یک فروبری طولپا صادق در شرط

آنگاه ، x در شرط $L_k x = Ax$ برای یک ماتریس A و یک عددی صحیح نامنفی $n < k$، صدق می‌کند اگر و تنها اگر M زیر مجموعه بازی از یکی از ابررویه های پایین باشد:

(آ) یک ابررویه با H_k ثابت و $H_{k+1} = 0$ ؛

(ب) $\mathbb{S}^m(\sqrt{1-r^2}) \times \mathbb{S}^{n-m}(r)$، با $0 < r < 1$ و $0 < m < n$، اگر $q = 0$؛

(پ) $\mathbb{H}^m(-\sqrt{r^2-1}) \times \mathbb{S}^{n-m}(r)$، با $r > 1$ و $0 < m < n$، اگر $q = 1$.

مراجع

[1] L. J. Alias, A. Ferández and P. Lucas, Submanifolds in pseudo-Euclidean spaces satisfying the condition $\Delta x = Ax + B$, *Geom. Dedicata* **42** (1992), no. 3, 345-354.

[2] L. J. Alias, A. Ferrández and P. Lucas, Hypersurfaces in space forms satisfying the condition $\Delta x = Ax + B$, Trans. Amer. *Math. Soc.* **347** (1995), no. 5, 1793-1801.

[3] L. J. Alias and S. M. B. Kashani, Hypersurfaces in space forms satisfying the condition $L_k x = Ax+b$, *Taiwanese J. Math* **14** (2010), no. 5, 1957-1977.

[4] T. Hasanis and T. Vlachos, Hypersurfaces of E_{n+1} satisfying $\Delta x = Ax + B$, *J. Austral.Math. Soc. Ser. A* **53** (1992), no. 3, 377-384.

[5] F . Pashaie and S. M. B. Kashani, Spacelike hypersurfaces in riemannian or lorentzian space forms satisfying $L_k x = Ax+b$, *Bull. math . Soc .Iranian* **18** (2013) 195-213.

[6] T. Takahashi, Minimal immersions of Riemannian manifolds,*J. Math. Soc. Japan* **18** (1966) 380-385.

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Invariant structures and gauge transformation of almost contact metric manifolds

Abstract: In this paper we found a condition such that a K-contact structure , a Sasakian structure and a cosymplectic structure be invariant under the gauge transformation.We have considered the gauge tranformation of 3-Sasakian manifolds, 3-almost contact manifolds and 3-cosymplectic manifolds with given conditions that they are invariant under gauge transformation. Finally we see that an slant submanifold of an almost contact manifold is invariant under the gauge transformation.

Keywords: Gauge transformation,almost contact manifold,3-almost contact manfold, 3-Sasakian manifold ,3-cosymplectic manifold.

1 INTRODUCTION

Gauge transformation of a contact metric manifold was introduced by Tanno in .In section 3 we take a condition that gauge transformation of a K-contact (sasakian) manifold carry a K-contact structure (resp sasakian).In section 4 we introduce the gauge transformation of 3-Sasakian manifolds. In section 5 we research about gauge transformation of Cosymplectic manifolds , almost contact 3-structures and 3-cosymplectic manifolds and we see that an slant submanifold of an almost contact manifold is invariant under the gauge transformation.

2 Preliminaries

An almost contact manifold is an odd-dimensional manifold M which carries a field φ of endomorphisms of the tangent spaces, a vector field ξ , called characteristic or Reeb vector field, and a 1-form η satisfying $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$, where$I : TM \longrightarrow TM$ is the identity mapping. From the definition it follows also that $\varphi\xi = 0$, $\eta o \varphi = 0$ and that the (1, 1)-tensor field φ has constant rank $2n$. An almost contact manifold (M, φ, ξ, η) is said to be normal when the tensor field $N = [\varphi, \varphi] + 2d\eta \otimes \xi$ vanishes identically,$[\varphi, \varphi]$ denoting the Nijenhuis tensor of φ. It is known that any almost contact manifold (M, φ, ξ, η) admits a Riemannian metric g such that

$$g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F) \qquad (1)$$

holds for all $E, F \in \Gamma(TM)$. This metric g is called a compatible metric and the manifold M together with the structure (φ, η, ξ, g) is called an almost contact metric manifold. As an immediate consequence of 1, one has $\eta = g(, \xi)$. The 2-form Φ on M defined by$\Phi(E, F) = g(E, \varphi F)$ is called the fundamental 2-form of the almost contact metric manifold M.A normal almost contact manifold that $d\eta = 0$ and $d\Phi = 0$ is called a cosymplectic manifold. Almost contact metric manifolds such

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that $d\eta = \Phi$ are called contact metric manifolds. Finally, a normal contact metric manifold is said to be a Sasakian manifold. A contact metric manifold is called K-contact if the tensor $h = \frac{1}{2}\pounds_\xi\varphi$ is vanishes.

An almost 3-contact manifold is a (4n + 3)-dimensional smooth manifold M endowed with three almost contact structures $(\varphi_1, \xi_1, \eta_1), (\varphi_2, \xi_2, \eta_2), (\varphi_3, \xi_3, \eta_3)$ satisfying the following relations, for every $\alpha, \beta \in \{1, 2, 3\}$,

$$\varphi_\alpha\varphi_\beta - \eta_\beta \otimes \xi_\alpha = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\varphi_\gamma - \delta_{\alpha\beta}I$$

$$\varphi_\alpha\xi_\beta = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\xi_\gamma, \quad \eta_\alpha o\varphi_\beta = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\eta_\gamma$$

$$\eta_\alpha(\xi_\beta) = 0 \tag{2}$$

where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric symbol. This notion was introduced by Kuo [4] and, independently, by Udriste[8]. In [4] Kuo proved that given an almost contact 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, there exists a Riemannian metric g compatible with each of them and hence we can speak of almost contact metric 3-structures. It is well known that in any almost 3-contact metric manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric g . Moreover, by putting $\mathcal{H} = \bigcap_{\alpha=1}^{3} ker\eta_\alpha$ one obtains a 4n-dimensional distribution on M and the tangent bundle splits as the orthogonal sum $TM = \mathcal{H}\oplus < \xi_1, \xi_2, \xi_3 >$. When the three structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ are contact metric structures, we say that M is a "3-contact metric manifold" and when they are Sasakian, that is when each structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$ is also normal, we call M a "3-Sasakian manifold". Also a manifold with an almost contact 3-structure is called a 3-cosymplectic manifold if each almost contact structure is a cosymplectic structure. Indeed as it has been proved in 2001 by Kashiwada [3], every contact metric 3-structure is 3-Sasakian.kuo [4] proved that there is a metric compatible with each structure.

Let $\tilde{M}$ be a submanifold immersed in M.We denote by $\tilde{g}$ the induced metric on $\tilde{M}$. Let $T\tilde{M}$ be the Lie algebra of vector fields in $\tilde{M}$ and $T\perp\tilde{M}$ the set of all vector fields normal to $\tilde{M}$.For any $X \in T\tilde{M}$, we write

$$\varphi X = TX + NX$$

where TX is the tangential components of the tangent bundle and N is a normal-bundle valued 1-form on the tangent bundle.according Lotta's definition , $\tilde{M}$ is slant if $\theta(X)$,the angle between φX and $T_x\tilde{M}$,is a constant which is independent of the choice of $x \in \tilde{M}$and$X \in T_x\tilde{M}- < \xi_x >$.

A gauge transformation of a contact metric manifold (M^{2n+1}, η, g) is a contact metric manifold $(M^{2n+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ defined by

$$\bar{\eta} = f\eta, \bar{\xi} = f^{-1}\xi + Z, \bar{\varphi} = \varphi - \bar{\eta}\otimes\varphi Z$$

$$\bar{g} = fg+f(f+f^2|Z|^2-1)\eta\otimes\eta-f^2\eta(Z,.)-f^2g(Z,.)\otimes\eta.$$

with $Z = \frac{1}{2}f^{-2}\varphi(\bigtriangledown f)$,for $f > 0$ on M

3 Gauge transformation of K-contact manifolds

In this section we take a condition that gauge transformation of a K-contact (resp Sasakian) manifold carry a K-contact (resp Sasakian)structure. In following theorem we find condition when gauge transformation of a sasakian manifold (K-contact resp) is sasakian (K-contact resp) too.

Theorem 3.1. *The gauge transformation of K-contact manifold is again K-contact manifold* (M, η, g) *iff* $(\pounds_Z\varphi)X = g(X, Z)\xi$ *for all* $X \in \chi(M)$.

Theorem 3.2. *Let* (M^{2n+1}, η, g) *be a sasakian manifold. The gauge transformation by* f *is also sasakian iff* $\varphi(\pounds_Z\varphi)X = 0$ *for all* $x \in \chi(M)$.

4 Gauge transformation of 3-sasakian manifolds

Now let $(M^{4n+3}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-sasakian manifold.If we put $\bar{\eta}_\alpha = f\eta_\alpha$ for $\alpha \in \{1,2,3\}$that f is a positive function on M, we have three new contact structures:

$$\bar{\eta_\alpha} = f\eta_\alpha, \bar{\xi_\alpha} = f^{-1}\xi_\alpha + Z_\alpha, \bar{\varphi_\alpha} = \varphi_\alpha - \bar{\eta_\alpha} \otimes \varphi_\alpha Z_\alpha$$

$$\bar{g_\alpha} = fg + f(f + f^2|Z_\alpha|^2 - 1)\eta_\alpha \otimes \eta_\alpha - f^2\eta_\alpha \otimes g(Z, .) - f^2 g(Z, .) \otimes \eta_\alpha.$$

with $Z_\alpha = \frac{1}{2}f^{-2}\varphi_\alpha(gradf)$.
These new contact metric structures have a 3-Sasakian structure if $gradf \in \mathcal{H}$.

Remark 4.1. *If we put $\bar{\eta_\alpha} = f_\alpha\eta_\alpha$,we will see from 2 that this transformation of a 3-sasakian structure can't carry a 3-sasakian structure unless $f_1 = f_2 = f_3$. Then we can use above calculations .*

Theorem 4.2. *Compatible metric with the new 3-Sasakian structure $(\bar{\eta}, \bar{\xi}, \bar{\varphi})$ is:*

$$\bar{g}(X,Y) = \bar{g_1}(X,Y) + \bar{g_2}(X,Y) + \bar{g_3}(X,Y) - 2fg(X,Y) \qquad (3)$$

So we can define the "gauge transformation of a 3-Sasakian manifold:

Definition 4.3. *If we put $\bar{\eta} = f\eta$ for $\alpha \in \{1,2,3\}$that f is a positive function on M and $gradf \in \mathcal{H}$, we say that the new 3-Sasakian structure*

$$\bar{\eta_\alpha} = f\eta_\alpha, \bar{\xi_\alpha} = f^{-1}\xi_\alpha + Z_\alpha, \bar{\varphi_\alpha} = \varphi_\alpha - \bar{\eta_\alpha} \otimes \varphi_\alpha Z_\alpha$$

$$\bar{g}(X,Y) = \bar{g_1}(X,Y) + \bar{g_2}(X,Y) + \bar{g_3}(X,Y) - 2fg(X,Y)$$

with $Z_\alpha = \frac{1}{2}f^{-2}\varphi_\alpha(gradf)$ is "gauge transformation" of a 3-sasakian manifold.

5 Gauge transformation of almost contact manifolds

Gauge transformation of an almost contact manifold is

$$\bar{\eta} = e^\sigma\eta, \quad d\bar{\eta} = e^\sigma(d\eta + d\eta \wedge \eta)$$

$$\bar{\xi} = e^{-\sigma}(\xi + \varphi A), \quad \bar{\varphi} = \varphi + \eta \otimes A$$

$$\bar{g} = e^{2\sigma}(g - \eta(\varphi A, .) - g(\varphi A, .) \otimes \eta + g(A, A)\eta \otimes \eta)$$

that $\sigma > 0$ and A is a vector field that $d(\varphi A, X) = d\sigma(hX)$ that hX is the projection of X in $ker\eta$.
A cosymplectic structure is invariant under this transformation iff $d\bar{\eta} = 0$ and $d\bar{\Phi} = 0$. With direct computation we have

so we have

Theorem 5.1. *A cosymplectic structure is invariant under the gauge transformation iff $d\sigma \wedge \Phi = 0$ and $d\sigma \wedge \eta = 0$.*

5.1 gauge transformation of a an almost contact 3-structure

Now we introduce "Gauge transformation of an almost contact 3-structure".Let $(M, \eta_\alpha, \xi_\alpha, \varphi_\alpha, g)$ with $\alpha \in \{1,2,3\}$ be a manifold with an almost contact 3-structure.If we put $\bar{\eta}_\alpha = e^\sigma\eta_\alpha$ then we have three almost contact structures

$$\bar{\eta}_\alpha = e^\sigma\eta_\alpha, \quad d\bar{\eta}_\alpha = e^\sigma(d\eta_\alpha + d\sigma \wedge \eta_\alpha)$$

$$\bar{\xi}_\alpha = e^{-\sigma}(\xi_\alpha + \varphi A_\alpha), \quad \bar{\varphi}_\alpha = \varphi_\alpha + \eta \otimes A_\alpha$$

$$\bar{g}_\alpha = e^{2\sigma}(g - \eta_\alpha \otimes g(\varphi A_\alpha, .) - g(\varphi A_\alpha, .) \otimes \eta_\alpha + g(A, A)\eta_\alpha \otimes \eta_\alpha)$$

With direct computation we can see that these almost contact structures have an almost contact 3-structures with each others iff $A_1 = A_2 = A_3$.Also we can see that compatible metric with this new almost contact 3-structure is

$$\bar{g} = \bar{g}_1 + \bar{g}_2 + \bar{g}_3 - 2e^{\sigma} g$$

So we can define

Definition 5.2. *Let $(M, \eta_\alpha, \xi_\alpha, \varphi_\alpha, g)$ with $\alpha \in \{1,2,3\}$ be a manifold with an almost contact 3-structure.If we put $\bar{\eta}_\alpha = e^{\sigma}\eta_\alpha$ that $\sigma > 0$ and $A_1 = A_2 = A_3$ then we have a new almost contact 3-structure*

$$\bar{\eta}_\alpha = e^{\sigma}\eta_\alpha, \quad d\bar{\eta}_\alpha = e^{\sigma}(d\eta_\alpha + d\sigma \wedge \eta_\alpha)$$

$$\bar{\xi}_\alpha = e^{-\sigma}(\xi_\alpha + \varphi A_\alpha), \quad \bar{\varphi}_\alpha = \varphi_\alpha + \eta \otimes A_\alpha$$

$$\bar{g} = \bar{g}_1 + \bar{g}_2 + \bar{g}_3 - 2e^{\sigma} g$$

We say that this new almost contact 3-structure is "Gauge transformation " the almost contact 3-structure $(\eta_\alpha, \xi_\alpha, \varphi_\alpha, g)$ with $\alpha \in \{1,2,3\}$.

Now with direct computation we have following theorem

Theorem 5.3. *Gauge transformaton of a 3-cosymplectic carry a 3-cosymplectic structure iff*

$$\forall \alpha \in \{1,2,3\}, \quad d\sigma \wedge \eta_\alpha = 0, \quad d\sigma \wedge \Phi_\alpha = 0$$

5.2 Slant submanifolds and gauge transformaton

Let $\tilde{M}$ be slant submanifold of the almost contact manifold $(M, \varphi, \eta, \xi, g)$ and $\theta(X)$ be the angle between φX and $T_x\tilde{M}$ .It can easily seen that the distribution $\mathcal{D}$ is invariant under the gauge transformation. So we can see $\bar{\theta}(X)$, that the angle between $\bar{\varphi}X$ and $T_x\tilde{M}$,is a constant which is independent of the choice of $x \in \tilde{M}$and$X \in T_x\tilde{M} - < \xi_x >$ and $\bar{\theta}(X) = \theta(X)$. So slant submanifolds of an almost contact manifold is invariant under the gauge transformation.

References

[1] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Birkhauser, Boston, 2002.

[2] G. Calvaruso, D. perrone, Contact pseudo-metric manifolds, Differential Geometry and its Applications 28 (2010)615-634.

[3] T. Kashiwada, On a contact 3-structure, Math. Z. 238 (2001) 829-832.

[4] Y.Y. Kuo, On almost contact 3-structure, Tohoku Math. J. 22 (1970) 325-332.

[5] S. sasaki, Y.Hatakeyama, On differrentiable manifolds with contact metric structures, J.Math. Soc. Japan 14 (1962) .

[6] S. sasaki, Y.Hatakeyama, On differrentiable manifolds with certain structures which are closely related to almost contact structures II, Thoku Math. J. 13 (1961) 281294.

[7] S. Tanno, The topology of contact Riemannian manifolds, Illois J. Math. 12 (1968) 700-717.

[8] C. Udriste, Structures presque coquaternioniennes, Bull. Math. Soc. Sci. Math. R. S. Roumanie 13 (1969) 487-507.

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Decomposition of Euclidean nearly Kähler submanifolds

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Abstract: In this article we study the foliation space of complex and invariant umblic distribution on isometric immersion from nearly Kähler manifold M into Euclidean space. We show that, under suitable condition this foliation space is a nearly Kähler manifold and M can be decomposed as the product space of a homogeneous 6-nearly Kähler manifold and this foliation space.
Keywords: Nearly Kähler, Isometric immersion, Foliation space, Intrinsic Hermitian connection .

1 INTRODUCTION

Concept of nearly Kähler manifold is related to the weak holonomy $U(n)$ introduced by A. Gray in 1971[6] and many interesting theorems about topology and geometry of nearly Kähler manifolds were proved by Gray using special equation on Riemannain curvature tensor like Kähler manifolds [5, 7]. Also nearly Kähler geometry naturally arises as one of the sixteen classes of almost Hermitian manifolds appearing in the celebrated Gray-Hervella classification [8].
One of the most important properties of nearly Kähler manifolds is that their instinct Hermitian connection is totally skew-symmetric and parallel torsion. From this point of view, they naturally fit into the setup proposed in [10] towards a weakening of the notion of Riemannian holonomy. The same property suggests that nearly Kähler manifolds might be objects of interest in string theory [9].

Maybe one of the important questions in geometry of nearly Kähler manifolds is Butrulle conjecture[2]:*Every complete (compact) nearly Kähler manifolds is 3-symmetric or equivalently is homogeneous.* According to the Nagy decomposition [1], this conjecture can be separated two restricted questions:

Problem 1. *Every complete (compact) 6-dimensional nearly Kähler manifolds is homogenous?*

Problem 2. *Every positive quaternion-Kähler manifolds is Wolf space?*

In [11, 3], we partially answered this conjecture by studing isometric immersions $f : M^{2n} \longrightarrow Q^{2n+p}$ from a nearly Kähler manifold into a space form (especially Euclidean space). We introduced in [11] an umbilic distribution which is complex and invariant by the torsion of intrinsic Hermitian con-

*Speaker

nection, we showed that, this distribution is integrable and each leaf of the generated foliation is a 6-dimensional homogeneous nearly kähler submanifold. Using this foliation we can parametrize isometric immersions from a nearly Kähler manifold into the Euclidean space and constract some examples of such submanifolds.[12, 4]

In this article, we focus on the foliation space of the above integrable distribution.

2 Main results

Definition 2.1. *The instinct Hermitian connection $\bar{\nabla}$ is the orthogonal projection of $\nabla \in \mathcal{SO}$ on $\mathcal{U}$. Equivalently, it is the unique Hermitian connection such that $\nabla - \bar{\nabla}$ is a 1-form with values in $u(M)^{\perp}$.*

The difference $\eta = \nabla - \bar{\nabla}$ is known explicitly, $\eta X = \frac{1}{2} J \circ (\nabla_X J)$ for all $X \in TM$, and measures the failure of the $U(n)$-structure to admit a torsion-free connection. It can be used to classify almost Hermitian manifolds. For example, Kähler manifolds are characterized by ∇ being a Hermitian connection: $\bar{\nabla} = \nabla$.

Definition 2.2. *Let M be an almost Hermitian manifold. The following conditions are equivalent and define a nearly Kähler manifold:*
(1) the torsion of $\bar{\nabla}$ is totally skew-symmetric,
(2) $(\nabla_X J)X = 0$ for all $X \in TM$,
(3) $\nabla_X \omega = \frac{1}{3} i_X d\omega$ for all $X \in TM$,
(4) $d\omega$ is of type $(0,3) + (3,0)$ and the Nijenhuis tensor N is totally skew-symmetric.

Definition 2.3. *Let $f : M^{2n}(<>, J) \longrightarrow Q^{2n+p}$ be an isometric immersion from a nearly Kähler manifold into a space form with second fundamental form α and $0 \neq \eta \in T_f^{\perp} M$ is a non-zero normal vector field on M. Umblic distribitaion of f defined by $\forall x \in M \quad x \mapsto \Delta_x$ where*

$$\Delta_x = \{X \in T_x M | \alpha(X,Y) =< X, Y > \eta \quad \forall Y \in T_x M\}$$

comlexification of this distribution described by $\Delta_x \cap \Delta_x' = \Delta_x \cap j\Delta_x$ where

$$\Delta_x' = \{X \in T_x M | \alpha(JX, Y) + \alpha(X, JY) = 0 \quad \forall Y \in T_x M\}$$

Now we put

$$\Delta_x'' = \{X \in T_x M | \alpha(T(X,Y), Z) + \alpha(X, T(Y,Z)) = 0 \quad \forall Y, Z \in T_x M\}$$

where T is the torsion of intrinsic Hermitian connection and specified by $T(X,Y) = (\nabla_x J)Y$.(∇ is Levi-Civita connection on M).
We define by $D_x = \Delta_x \cap \Delta_x' \cap \Delta_x''$ umblic distribution which is complex and invariant by the torsion of intrinsic Hermitian connection.

It is easy to show that

$$D_x = \{X \in T_x M | X, JX, T(X,Y) \in \Delta_x \quad \forall Y \in T_x M\}$$

For a "saturated" open connected subset $U \in M$ (meaning each leaf of complex and invariant umbilic foliation in U is maximal in M) we consider the quotient space U/D of leaves in U with the projection map $\pi : U^{2n} \longrightarrow V^{2n-6} = U^{2n}/D$. In general case V is not a manifold. (it could fail to be Hausdorff). But if each leaf of the complex and invariant umbilic foliation in U^{2n} is complete then V becomes a manifold.[4]. V is called the foliation space of complex and invariant umbilic foliation.

Theorem 2.4. *Each leaf of complex and invariant umbilic foliation N in M is minimal and second fundamental form of each leaf is formed by $\forall X, Y \in \chi(N); \beta(X, JY) = J\beta(X,Y)$. Also $\eta = 3H$ where H is the mean curvature of $N^6 \hookrightarrow M^{2n} \longrightarrow Q_c^{2n+p}$ and $c + ||\eta|| = \frac{S}{30}$ (S is scalar curvature of each leaf)*
(S is constant for each leaf because each leaf is 6-nearly Käher manifold and therefore is Einstein)

Theorem 2.5. *Let $f : M^{2n} \longrightarrow R^{2n+p}$ be an isometric immersion from a nearly Kähler manifold*

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into Euclidean space. If each leaf of the complex and invariant umbilic foliation is complete then leaf space is an almost complex manifold such that top cohomological group of this space in non-trivial. When V (leaf space) is compact and M is complete with chosing a suitable metric on V, the projective map $\pi : M \longrightarrow V$ is Riemannian submersion such that $\pi \circ J^M = J^V \circ M$
(this means that π is an almost Hermitian submersion from nearly Kähler manifold M)
therefore V is almost Hermitian and finally is a nearly Kähler manifold.

Theorem 2.6. *Let $f : M^{2n} \longrightarrow R^{2n+p}$ be an isometric immersion from a nearly Kähler manifold into Euclidean space. Orthogonal complement of complex and invariant umbilic distribution with respect to a Riemannian metric is an integrable distribution and each integral submanifold of this distribution is totally geodesic in manifold M.*

Corollary 2.7. *Let $f : M^{2n} \longrightarrow R^{2n+p}$ be an isometric immersion from a complete and simply connected nearly Kähler manifold into Euclidean space. on manifold M there exist two integrable distributions $D, D^{\perp}$ such that each leaf of generated foliations by D and $D^{\perp}$ is minimal and totally geodesic in M respectively. Moreover, if $\forall X \in D^{\perp}_x, U \in D_x$ we have*

$$R(X, JX, U, JU) = 0 \qquad (1)$$

then M is product space of the foliation space of complex and invariant umbilic foliation and a 6-dimensional homogeneous nearly Kähler manifold which is isometric with the corresponding factor in Nagy decomposition.

Remark 2.8. *Note that 6-dimensional factor in Nagy decomposition is not necessarily homogeneous therefore this is a positive answer to the problem 1 under assumption of Corollary 2.7.*

References

[1] P.-A. Nagy, "Nearly Kähler geometry and Riemannain foliations," Asain J. Math, vol. 6, pp.4811-504, 2002.

[2] J.-B. Butruille, "Homogenous Nearly Kähler manifolds"Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lect. Math. theor. Phys European Math. Soc. publishing House, Zurich,pp.399-423, 2010

[3] A. Heydari and N. Heidari, "Nearly Kähler submanifolds of a space form," submitted 2013.

[4] A. Heydari and N. Heidari, "parametrization of Nearly Kähler submanifolds of a standard space form," preprint 2014.

[5] A. Gray, " Nearly Kähler manifolds" J. Diff. Geom. Vol 4, pp. 283-309,(1970)

[6] A. Gray," Weak holonomy groups", Math. Z, Vol 125 pp. 290-300, 1971

[7] A. Gray, "The structure of nearly Kähler manifolds", Math. Ann, Vol 223, pp. 233-248. 1979

[8] A. Gray and L. M. Hervella, "The sixteen classes of almost Hermitian manifolds and their linear invariants", Ann. Math. Pure Appl, Vol .23, pp. 35-58 1980

[9] Th. Friedrich and S. Ivanov," Parallel spinors and connections with skew-symmetric torsion in string theory", Asian J. Math,Vol 6 , pp.303-336, 2002

[10] R. Cleyton, A. Swann," Einstein metrics via intrinsic or parallel torsion", Math. Z, Vol.247, pp 513-528, 2004

Papers from Conference Proceedings (Published):

[11] N. Heidari and A. Heydari "6-Nearly Kähler hypersurfaces in space forms," in Proc. of Conference (the 43rd Annual Iranian Mathematics Conference) pp. 394-397, Tabriz University, Tabriz Iran 2012

[12] N. Heidari and A. Heydari "Nearly Kähler hypersurface of a Space form," in Proc. of Conference (the 44rd Annual Iranian Mathematics Conference) pp.799-802, Ferdowsi University of Mashhad, Iran, 2013

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Reconstructing of the causal structure of the space-time

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One of the most important problem in the Einstein's theory of the relativity is the causality. The causal curves connecting points of the space-time determine the causal structure and have a fundamental role in the study of the structures of space-times. Much of information of causal structure of space-times is corresponded to the topological properties of the space of causal curves. Furthermore, by the space of null curves and a family of skies of the events in the space-time, structures of the space-time can be reconstructed.
Keywords: Lorentzian geometry, Causal structure of the space-time, Null geodesic.

1 INTRODUCTION

In the theory of the general relativity, a space-time is a smooth differentiable manifold M equipped with a smooth Lorentz metric g. A path is a continuous map $\Gamma : I \to M$, where I is a connected open subset of $\mathbf{R}$. The term "curve" will be used either for the image of such a map or (more correctly) for an equivalence class of paths up to a parameter change. In the space-time, the causal relations of two events are depended on the set of the causal curves which connects these to each other. Using the space of causal curves of a space-time rather than the space-time itself provides a surprisingly simpler framework for the study of the causal structure. The space of null geodesics can be considered as a subspace of the topological space of causal curves and the causal structure of a space-time is uniquely determined by the space of its null geodesics. Indeed, the conformal structure of a space-time can be reconstructed from the topology and geometry of its space of future oriented, unparametrized null geodesics $\mathcal{N}$[1]. So, space-times might be characterized by either the topological properties or the smoothness of the space of their null geodesics. There are some studies on this idea R. J. Low described how the information on the various aspects of the causal structure of a space-time can be stated in terms of the properties of the corresponding space of null geodesics [4]. Similar incidences can be seen in [3], [5] and [6] in which the geometry and topology of the space of null geodesics and the space of causal curves are proven to be more helpful than the underlying space-time. For example, the T_2-separation axiom of the space of causal geodesics of a space-time determines the existence of the naked singularities of the space-time.

In this paper, we attempt to investigate some relations between the causal structure of a space-

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time M and separation axioms of the topological space $C(M)$ of its causal curves. Our study is based on the separation axioms of the τ^0-topology on $C(M)$ introduced in [5]. Here, the problem of reconstructing a space-time M from $\mathcal{N}$ will be addressed. In this problem a paramount role is played by the space of skies $\sum$ of the space-time M where the sky $S(p)$ of a given point $p \in M$ is the congruence of light rays passing through it. Low's conjecture that states that two events in a time-oriented Lorentzian manifold are causally related iff their corresponding skies, which are legendrian knots with respect to the canonical contact structure in the space of null geodesics,are linked, constitutes one of its most salient outcomes [2]. Thus the exploration of the relation between the causal properties of a conformal class of Lorentzian metrics and the topological properties of skies in the manifold of null geodesics opens a new and exciting relation between the topology and causality relations of Lorentzian space-times and the topology of contact manifolds.

2 The space of causal curves

First, we recall the τ^0-topology on $C(M)$ defined in [5]. A member of the sub-basis of the desired topology consists of the class of all causal curves $\gamma \in C(M)$ passing through an open subset U of M.

Definition 2.1. *A curve γ in M is said to be a "somewhere dense curve" if it is a somewhere dense subset of M, i.e. its closure has a nonempty interior.*
A somewhere dense causal curve in M is a causal curve which is somewhere dense in M.

We obtain some conditions for which $C(M)$ is T_1-topological space. In fact, we prove the existence of an inseparable pair of causal curves; that is, we prove the existence of a pair of causal curves such that at least one of them is in the closure of the other. If exactly one of these curves has this property, T_1-axiom is violated (but T_0-axiom may still hold) and it occurs, if M contains a somewhere dense causal curve. If each of these curves lies in the closure of the other, T_0-axiom is violated and we shall investigate this case later.

Theorem 2.2. *If there exists a causal curve which is dense in an open subset of M, then τ^0 fails to satisfy the T_1-axiom.*

Example 2.3. *Consider the two dimensional space-time M_2 given by the quotient of two-dimensional Minkowski space under the action of $Z \times Z$ given by $(m, n).(x, t) = (x + m\sqrt{2}, t + n)$. Each null geodesic of this space is dense in M_2, and any neighborhood of a rightward moving null geodesic contains every other rightward moving null geodesic. Thus $C(M_2)$ is not T_1.*

In the following theorem, we find a condition for a space-time M to guarantee the existence of a somewhere dense causal curve in M.

Definition 2.4. *A causal curve is said to be a "proper causal curve" if it is not a null curve.*

Theorem 2.5. *The space-time M contains a somewhere dense causal curve if and only if M has a proper causal closed curve.*

Corollary 2.6. *The space-time M contains a somewhere dense causal curve if and only if it contains a timelike closed curve.*

We are now in the position to state and prove the following result, which provides a sufficient condition for the T_0-separation axiom of the τ^0-topology.

Theorem 2.7. *If the space $C(M)$ of causal curves does not satisfy in T_0-axiom, then there exists a somewhere dense causal curve in M.*

Corollary 2.8. *If M is a chronological space-time, then $C(M)$ is a T_0-space.*

Theorem 2.9. *If M is a distinguishing space-time, then $C(M)$ is a T_1-space.*

The null geodesics are fundamental to the causal structure of space-time M. Motivated by this observation, we can consider the space of all null geodesics as a subspace of topological space of $C(M)$ and investigate the relationship between its topology and the causal structure of M.

Corollary 2.10. *If M is a distinguishing space-time, then the space of null geodesics of M satisfies the T_1-separation axiom.*

We have seen that the existence of a proper causal closed curve or, equivalently, the existence of a somewhere dense causal curve in M violates the T_1-separation axiom of the τ^0-topology. Hence, a model for a space-time is preferred to be without a causal somewhere dense curve.

3 The space of null geodesics

Let us denote by TM the tangent bundle of M and by $\pi_M : TM \to M$ the corresponding canonical projection.
The set $N^+ = \{\xi \in TM : \mathbf{g}(\xi,\xi) = 0, \xi \neq 0, \xi \text{ future}\} \subset TM$ defines the sub bundle of future null vectors over M. Any element $\xi \in N^+$ defines a unique future oriented null geodesic γ in M such that $\gamma(0) = \pi_M(\xi)$ and $\gamma'(0) = \xi$. Consider the quotient space of N^+ with respect to positive scale transformations, i.e., the quotient space with respect to the dilation, or Euler vector field Δ on N^+, that is the space of leaves of the vector field whose flow is given by $e^t\xi$, $t \in R$. In this way, we obtain the bundle PN^+ of future null directions

$$PN^+ = \{[\xi] : u \in [\xi] \Leftrightarrow\}u = \lambda\xi \ , 0 \neq \lambda \in R^+, \xi \in N\}$$

Now, any $[\xi] \in PN^+$ defines an unparametrized future oriented null geodesic, i.e., a light ray, in M which is the image in M of the null geodesic γ defined by $\xi \in N^+$. We denote by $\pi\colon P^+ \to M$ the canonical projection of the bundle PN^+ over M. The fibre $\pi^{-1}(p)$ is diffeomorphic to the standard sphere S^{m-2}. We observe that the bundle PN^+ is foliated by the lifts of these light rays to PN^+, which are projections to PN^+ of integral curves of the geodesic spray X_g restricted to N^+. We will call F to this foliation. Then, the space of null geodesics $\mathcal{N}$ can be defined too as the quotient space PN^+/F or, equivalently, as the quotient space of N^+ by the foliation K whose leaves are the maximal integral sub manifolds lying in N^+ of the integrable distribution defined by Δ and X_g, this is $\mathcal{N} \cong PN^+/F = N^+/K$. We will denote by σ the canonical projection $\sigma\colon PN^+ \to PN^+/F$ [1].

The quotient space PN^+/F is not a differentiable manifold in general. The next Proposition that guarantee that $\mathcal{N}$ inherits a differentiable structure.

Theorem 3.1. *Let M be a strongly causal space-time of dimension m. Then PN^+/F inherits a canonical differentiable structure from PN^+ of dimension $2m-3$ such that σ is a smooth submersion. Moreover, if M is not nakedly singular, then PN^+/F is Hausdorff.*

Hence, for any strongly causal space–time M without naked singularities, the space of null geodesics $\mathcal{N}$ inherits the structure of a Hausdorff smooth $(2m-3)$-dimensional differentiable manifold via the natural identification of $\mathcal{N}$ with PN^+/F and $\sigma : PN^+ \to \mathcal{N}$ is a submersion. Thus in what follows we will assume that M is a strongly causal not nakedly singular space-time and we call the space of null geodesics $\mathcal{N}$ equipped with the smooth structure above.

Definition 3.2. *Given a point $p \in M$, the set of null geodesic passing through p will be called "the sky of" p and it will be denoted by $S(p)$, i.e.*

$$S(p) = \{\gamma \in \mathcal{N} : p \in \gamma \subset M\}.$$

Notice that the geodesics $\gamma \in S(p)$ are in one–to–one correspondence with the elements in the fiber $\pi^{-1}(p) \subset PN^+$, hence the sky $S(p)$ of any point $p \in M$ is diffeomorphic to the standard sphere S^{m-2}. Now, it is possible to define the "space of skies" as

$$\Sigma = \{X \subset \mathcal{N} : X = S(p) \text{ for some } p \in M\}$$

and the sky map *as the application* $S : M \to \Sigma$ *that maps every* p *to* $S(p) \in \Sigma$. *This sky map* S *is, by definition of* Σ*, surjective. If the sky map* S *is a bijection, its inverse map denoted by* $P = S^{-1} : \Sigma \to M$ *will be called the "parachute map".*

4 Recovering of the Space-Time

A natural question to ask is how the causal structure of M is reflected in $\mathcal{N}$. We equip Σ with topological and smooth structure and then recover space-time by this space. We will start by defining a natural topology on the space of skies Σ induced by the topology of $\mathcal{N}$.

Lemma 4.1. *The collection of sets* $T = \{U \subset \Sigma : \widetilde{U} = \bigcup_{X \in U} X$ *is open in* $\mathcal{N}\}$ *is a topology in* Σ.

The next theorem give a smooth structure to the space of skies and reconstruct the smooth structure of the space-time.

Theorem 4.2. *Let* $V \subset M$ *be a globally hyperbolic convex normal open set such that* $U = S(V) \subset \Sigma$ *is a regular open set. Then* U *has a canonical differentiable structure depending only on* $\mathcal{N}$. *Moreover, There exists a unique differentiable structure in* Σ *compatible with the differentiable structure of* $U \subset \Sigma$ *such that the sky map* $S\colon M \to \Sigma$ *and the parachute map* $P\colon \Sigma \to M$ *are diffeomorphisms.*

Definition 4.3. *Let* (M, C) *and* $(\overline{M}, \overline{C})$ *be two strongly causal manifolds and* $(\mathcal{N}, \Sigma)$ *and* $(\overline{\mathcal{N}}, \overline{\Sigma})$ *theirs corresponding pairs of spaces of null geodesics and skies. We say that a map* $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ *"preserves skies" if* $\phi(X) \in \overline{\Sigma}$ *for any* $X \in \Sigma$.

Moreover, (M, C) *is said to be "recoverable" if for* $(\overline{\mathcal{N}}, \overline{\Sigma})$ *corresponding to* $(\overline{M}, \overline{C})$ *another strongly causal manifolds and* $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ *a diffeomorphism preserving skies,then the map*

$$\varphi = \overline{P} \circ \phi \circ S : M \to \overline{M}$$

is a conformal diffeomorphism on its image, where $\overline{P} : \overline{\Sigma} \to \overline{M}$ *is the parachute map to* $\overline{M}$.

Theorem 4.4. *Let* (M, C) *be a strongly causal manifold, then* M *is recoverable.*

References

[1] A. Bautista, A. Ibort, J. Lafuente, "On the space of light rays of a space-time and a reconstruction theorem by Low," Class. Quantum Grav. **31**, 075020 (2014).

[2] V. Chernov, Yu. Rudyak, "Linking and causality in globally hyperbolic space-times," Comm. Math. Phys. 279 (2008), 309-354.

[3] R. J. Low, "Causal relations and spaces of null geodesics, PhD Thesis," Oxford University (1988).

[4] R. J. Low, "The geometry of the space of null geodesics," J. Math. Phys. 30(4) (1989), 809-811.

[5] R. J. Low, "The space of null geodesics (and a new causal boundary)," Lecture Notes in Physics 692, Springer, Berlin Heidelberg New York, 2006, pp. 35-50.

[6] R. Pourkhandani, Y. Bahrampour, "The space of causal curves and separation axioms," Class. Quantum Grav. **29**, 015014 (2012).